

Analytic Tomography

This book is about tomography, which is a way to see what is inside an object without opening it up. The unifying idea of tomography is the Radon transform, which is introduced in an informal and graphic way in Chapter 1. The remaining chapters deal with the basic and advanced properties of the Radon transform and related operators.

The book was written to appeal to the broadest possible group of readers. The first chapter, which introduces computerized tomography, x-ray imaging and the Radon transform, requires almost no mathematical background. The second chapter, which is devoted to a rigorous and detailed study of the basic properties of the Radon transform should be accessible to readers with a good undergraduate background in mathematics. The last three chapters are devoted to the more advanced areas of mathematical tomography and the Radon transform. These chapters require a more sophisticated background in mathematics. There are numerous figures and more than 600 references to literature in the field.

Andrew Markoe is Professor of Mathematics at Rider University. He is the author of 19 publications in the areas of several complex variables, Radon transforms, and mathematical tomography.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Analytic Tomography

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Introduction

This book is about tomography, which is a way to see what is inside an object without opening it up. If you are intrigued with this idea, then, no matter what your background, you will find that at least some portion of this book will provide interesting reading. If this idea is not intriguing, then I would recommend some other publication for your reading pleasure.

The unifying idea of tomography is the Radon transform, which is introduced in an informal and graphic way in chapter 1. Reading chapter 1 will give you a good idea of the precise meaning of tomography. Reading chapter 2 will give you a very good idea of the meaning of tomography and if you read the last few chapters you will have a really good understanding of this idea. However, some of the later chapters will only be accessible to specialists.

I tried to write this book with two main ideas in mind. I wanted it to appeal to the broadest possible group of readers and I wanted it to be as comprehensive as possible. Therefore, chapter 1 has almost no mathematics in it – at least it does not require the reader to have any background beyond a good course in secondary school mathematics. CT (computerized tomography) scanners are used for medical diagnosis and produce detailed pictures of the human anatomy without opening up the patient. The dedicated reader will learn, in a very graphic way, how a CT scanner works. I hope this chapter will also be interesting to specialists who will see how the Radon transform and integral convolutions correspond to some familiar everyday processes.

Chapter 2 presupposes some knowledge of calculus. I tried to write it so that a second-year undergraduate student in mathematics or the sciences would have enough background to read the chapter. However, most readers of this chapter should have a few more mathematics courses beyond the elementary calculus sequence. This chapter rigorously introduces the Radon transform which is the basis of the rest of the book. I was somewhat surprised, myself, to see that almost all the basic theory of the Radon transform could be developed with not much more than the change of variables formula for integrals and Fubini's theorem on multiple integration. Students in the basic calculus sequence should know these formulas, at least in dimension two. However, and here

is where it requires some dedication, I develop the theory in n dimensions, explaining the necessary generalizations as the chapter proceeds.

Chapter 3 requires at least some graduate-level mathematics. This is where we generalize the Radon transform to the k -plane transform. The resulting analysis is much deeper than that of the first two chapters. The minimal requirements are a knowledge of real analysis (at least through the general theory of Lebesgue measure and integration and some elementary Fourier analysis) and a minimal familiarity with group theory. Readers having this sort of background should include graduate students in mathematics, even just after the first year, and many scientists and engineers. There are many more ideas necessary to understand this chapter, but I have taken care to at least explain the notation and basic concepts and to provide references for the interested reader whose background does not reach this far. There is no reason to expect that even a mathematician who is not a full-time analyst would have a good enough working knowledge of Grassmann manifolds, Haar measure, and distribution theory to really understand this chapter. Therefore, I have tried to err on the side of providing more detail, even if some ideas and arguments could be made more succinct when aimed at a specialist in the field. This will probably annoy most of my colleagues, and for this I hastily offer an apology. However, I think other readers will be thankful for the amount of detail that I have provided.

The material in chapter 3 is essentially self-contained beyond the prerequisites that I just mentioned. Any more advanced ideas are described carefully enough that a mathematically literate reader should be able to follow the arguments, although considerable effort may be required. Except for a set of measure zero, all proofs in this chapter are self-contained.

However, in the remaining chapters I do not always provide full proofs. In general, in these chapters I present some basic ideas with full details and then I expand on these ideas. But I do not always give full, or sometimes even any, proofs. When proofs are omitted I always provide appropriate references to the literature. Therefore, in Chapters 2 and 3 when “Theorem . . .” occurs, you can almost always be sure to find “Proof. . . ■” immediately following.¹ However, in subsequent chapters you may find some theorems without proofs. Sometimes I mention this, but, in general, the lack of a proof indicates that the demonstration may be found in the associated reference.

I have provided a brief summary of prerequisites in the introduction to each chapter.

Here are the exceptions to the basic policy that I have just outlined. In chapter 1 there is a technical note that requires knowledge of some elementary calculus. In chapter 3, section 3.10, there are some very interesting results on how the k -plane transform acts on L^p functions. These results require a much more extensive development of the Riesz potential than I was able to provide. In fact, I probably would have required another volume just to provide these prerequisites. Therefore, I took the liberty of stating the main ideas and results about Riesz potentials without proof but with references.

I have tried to make this book a comprehensive treatment of the subject of analytic methods in tomography. However, it was impossible to go into full detail concerning

¹ The symbol ■ denotes the end of a proof.

every aspect of this field. Therefore, each chapter has a section titled “Additional References and Results” which can be thought of as a guide to the literature for the reader who wants to delve more deeply into the ideas of the chapter. Most of the material in these sections consists of a reference to the literature with a brief description of the author’s contribution. You will occasionally find more detail. Also, in these sections, you will occasionally find historical comments. I am not a historian, but I have tried to make these comments as accurate as possible.

Some topics could, maybe even should, be in this volume, but because of space and time restrictions they were not included. One example that comes immediately to mind is the area of impedance tomography. This area is extremely interesting, valuable, and analytic in nature. However, the background necessary to understand this exceeds both what I expect of the “generic” reader and also my ability to fit this theory into the number of pages and amount of time available to me. Similarly I could only give a brief introduction to the relation between tomography and partial differential equations, twistors, several complex variables and $\mathcal{D}$ modules. I hope that my readers will see that any field not included or only briefly treated is of the nature that would probably require a volume all by itself. However, I have tried to make at least passing mention of any area of tomography that is at all related to analysis.

Tomography, which may be justifiably defined as the study of the Radon transform, is itself part of the field of integral geometry. Tomography is divided into roughly two fields: geometric tomography and analytic tomography. Although geometric tomography is mostly concerned with probing the interior structure of geometric objects by using the techniques of geometry, analytic tomography has the same aim but uses techniques that are intimately related to both classical and modern real analysis, and sometimes also to complex analysis. These techniques include Lebesgue integration, the theory of distributions, and Fourier analysis.

Geometric tomography is treated in the excellent book by Gardner [185], but not at all by me. There are also several texts that deal with the analytic aspects of tomography (see section 2.9.2). There is some overlap between this volume and these other texts, but I believe that this volume has a unique emphasis, choice of topics, and point of view.

Boris Rubin has proposed writing a book with the tentative title “Introduction to Radon Transforms: Real Variable Methods, Integral Geometry and Harmonic Analysis.” This work will treat in much more detail some of the more advanced topics of this volume, in particular, those dealing with singular and fractional integration.

A search of the mathematical literature since 1917² that has a relation to tomography or the Radon transform will yield well over 2,000 publications. This does not take into account publications in other fields such as medicine, physics, and engineering. I have no idea what the total number of papers on tomography is, but I would not be surprised if it is greater than 10,000. I found about 1,500 papers that might even be remotely

² 1917 is the year of publication of Johann Radon’s ground-breaking paper [508]. Although one can trace the origins of tomography further back, the year 1917 is generally believed to signal the beginning of tomography.

related to this book. Because of the time and page restraints, I had to narrow this down even more. The result is the set of papers that you see in my bibliography. This is a substantial set of references, but clearly it is not exhaustive. Therefore, I apologize to any colleagues who were not mentioned or who received only a passing mention. This set includes many colleagues who work on the more applied area of the subject. Their work is interesting and valuable, but unfortunately I could not include every possible reference in this book.

The reader should be aware that this is a book on pure mathematics. This is inevitable because mathematical analysis depends on infinite processes, whereas any applied mathematical problem eventually has to deal with a finite process. Not many CT scanners exist that need to handle objects defined by general L^p functions. So, if you are planning to build a CT scanner in your garage, this book is not going to tell you how to do it. However, I believe that many engineers, physicists, and applied mathematicians will benefit from the theory that is presented here. For those of you who do want to build a CT scanner in your garage, after you read this book, you should look into Herman [296], Kak and Slaney [328], Natterer and Wübbeling [446], and Epstein [150].

Tomographers tend to be split into two, probably disjoint, classes. Members of the first class believe that nothing practical can come out of a theorem depending on, say, infinitely many x-ray projections. The other class, of which I am a member, disagrees. However, because I respect the opinions of the first class, I therefore abandon all pretense of presenting practical applications, although in a few places I make some remarks heading in that direction. This frees me to present the general theory, which happens to be a beautiful mathematical gem.

Dedication and Acknowledgments

I dedicate this book to my wife Ruth and to my children Ariana, Abigail, and Emily. It is written in honor of my mother Hyacinth Markoe and in loving memory of my father Ralph Markoe, my wife's parents Charles and Rachel Kalisky and my brother-in-law Henry Jones.

I had an enormous amount of help while writing this book. I thank my wife Ruth Markoe for her support and love and for putting up with me during this project. I also thank my children for the same reasons.

I am grateful for the support provided by Rider University in the form of research leaves and financial grants for this project. I am particularly grateful for the help I received from the interlibrary loan department of the Moore Library of Rider University.

The Institute for Advanced Study was gracious in appointing me Director's Visitor for the Spring of 2002. This gave me the opportunity to pursue this research in a very pleasant and productive environment. I thank Phillip Griffiths, the director of the Institute at that time and also Momota Ganguli and Judith Wilson-Smith, of the Institute's library, and Kate Monohan and Linda Geraci of the Director's office.

I thank the Institute of Physics Publishing, in particular, Elaine Longden-Chapman and Lara Finan, for arranging access to back issues of their journal *Inverse Problems*.

I thank the Siemens Corporation for providing images of CT scans and CT scanners.

I thank the Staff at Cambridge University Press and at TechBooks, Inc. for their efforts on behalf of producing this book. I am especially grateful to Jessica Farris of Cambridge University Press and the anonymous copy editor at TechBooks who had to deal with many fatuous errors on my behalf.

My gratitude and a salute go to my students Harry Doctor and Sharon Kobrin who proved that the first two chapters of this book could be read by undergraduates. Also my thanks go to them for help in proofreading those chapters.

Finally, I thank my many colleagues who helped me while I was writing this volume and from whom I learned so much. In particular, I thank Mark Agranovsky, Anthony Bahri, Jan Boman, Rolf ClackDoyle, Richard Gardner, Fulton Gonzalez, Eric Grinberg, Gabor Herman, Sigurdur Helgason, Alexander Katsevich, Fritz Keinert, Peter Kuchment, Rob Lewitt, Erwin Lutwak, Eric Todd Quinto, Boris Rubin, and Elias Stein.

Computerized Tomography, X-rays, and the Radon Transform

1.1 Introduction

The purpose of this chapter is to give an informal introduction to the subject of tomography. There are very few mathematical requirements for this chapter, so readers who are not specialists in the field, indeed who are not mathematicians or scientists, should find this material accessible and interesting. Specialists will find a graphic and intuitive presentation of the Radon transform and its approximate inversion.

Tomography is concerned with solving problems such as the following. Suppose that we are given an object but can only see its surface. Could we determine the nature of the object without cutting it open? In 1917 an Austrian mathematician named Johann Radon showed that this could be done provided the total density of every line through the object were known.¹ We can think of the density of an object at a specific point as the amount of material comprising the object at that point. The total density along a line is the sum of the individual densities or amounts of material.

In 1895 Wilhelm Roentgen discovered x-rays, a property of which is their determining of the total density of an object along their line of travel. For this reason, mathematicians call the total density an *x-ray projection*. It is immaterial whether the x-ray projection was obtained via x-rays or by some other method; we still call the resulting total density an x-ray projection.

Combining Roentgen's x-rays with Radon's idea gives a way of determining an unknown object without cutting it open. We call this process *tomography*.

Tomography can be applied to any object for which we can determine the x-ray projections either by actual x-rays or some other method. Tomography is used to investigate the interior structure of the following objects: the human body, rocket motors, rocks, the sun (microwaves were used here rather than x-rays), snow packs on the Alps, and violins and other bowed instruments. This list could be expanded to hundreds of objects. In this chapter we will see how tomography can be used to obtain detailed information about the human brain from its x-ray projections.

¹ Johann Radon (1887–1956) published the first discussion and solution of a tomographic problem (see reference [508] in the bibliography).

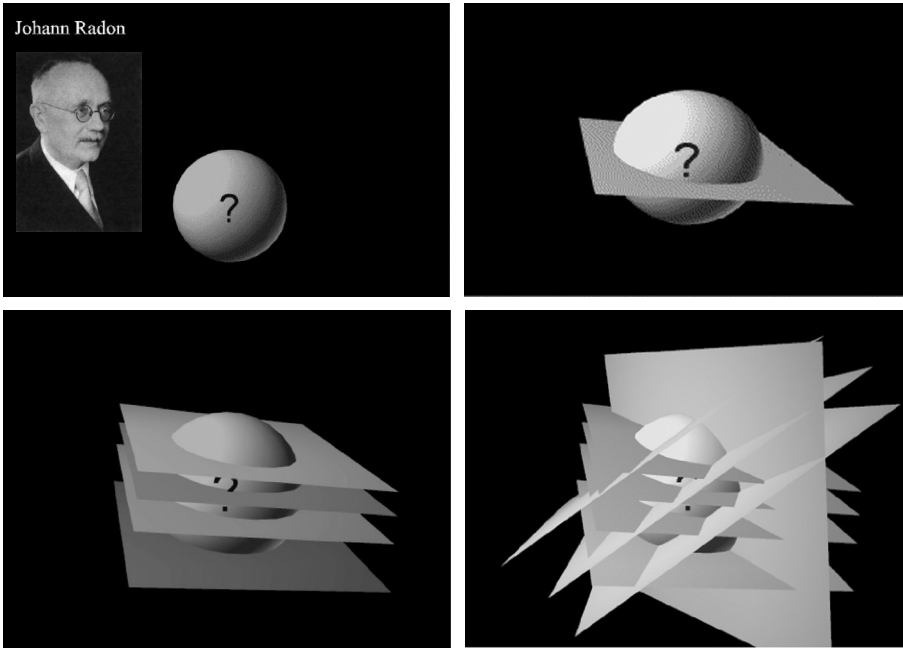


Figure 1.1. Johann Radon tries to figure out what is inside the sphere.

Allan M. Cormack and Godfrey N. Hounsfield shared the 1979 Nobel Prize in Physiology and Medicine for their contributions to the medical applications of tomography. The reference to snow packs comes from Cormack's Nobel prize lecture in Stockholm in 1979 (compare [102]).² Cormack remarked that the publication of his ideas on tomography took place in 1963 and 1964 and that "*There was virtually no response. The most interesting request for a reprint came from the Swiss Centre for Avalanche Research. The method would work for deposits of snow on mountains if one could get either the detector or the source into the mountain under the snow!*"

Radon not only showed how to determine a plane object from lines, but he also showed how to determine a solid object by using planes. We can visualize the discussion up to this point. In figure 1.1 Johann Radon is pondering what is inside the spherical object. In the next scene he decides to compute the total density on a single plane through the sphere. He knows that this is not enough information to determine the object, so he successively intersects with more and more planes. When he has collected the densities on all planes, then he is able to determine the object. How this may be done by using lines through a two-dimensional object is the subject of the remainder of this chapter. You do not need much background in mathematics to read this chapter – some knowledge about triangles and the ability to read a graph is really all that is required.

² Numbers in square brackets correspond to the list of references at the end of the book. For example, [102] refers to the article by Cormack that is listed in the references section.

1.2 Computerized Tomography (CT) and Mathematical Tomography – “Now, suddenly, the fog had cleared”

The Greek word $\tau\omicron\mu\omicron\sigma$ (tomos), meaning slice, is the source of the term *tomography*. This term was first used in diagnostic medicine. Since the discovery of x-rays by Roentgen, diagnosticians have attempted to produce images of human organs without the blurring and overlap of tissue that occurs in traditional x-ray pictures, such as the x-ray of the skull in the accompanying figure.



Courtesy of Ass. Prof. Dr. Mircea-Constantin Sora, MD, Ph. D., Medical University of Vienna.

We will see that tomography can produce much more detailed pictures from x-ray data. The reference to the fog clearing in the title of this section is from the presentation speech for the 1979 Nobel prize for Physiology or Medicine which was awarded, jointly to A. M. Cormack and G. N. Hounsfield in 1979. The presentation speech containing the preceding quotation was delivered by Professor Torgny Greitz of the Karolinska Medico-Chirurgical Institute and it is interesting to read the excerpts from this speech in Section 1.10.1.

Computerized Tomography, also known as **CT**, refers to the actual process of producing a detailed picture of the interior of an organism by using x-rays. **Mathematical tomography** refers to the mathematical process by which the picture is obtained. Computerized tomography is accomplished by designing a machine consisting of x-ray sources and x-ray detectors combined with a computer. The computer uses an algorithm adapted from the field of mathematical tomography to combine the data obtained from the x-rays into a detailed picture of the organs and tissue in a specific slice of a



Figure 1.2. A typical CT scanner. This one is manufactured by the Siemens Corporation. Courtesy of the Siemens corporation.

patient's body. This type of machine is called a **CT scanner**.³ A CT scanner can produce a clear and detailed image, called a **tomogram**, of the interior of a human body. This is done without cutting open the body, merely by sending x-rays through the tissue in question. How this is done is explained later in this chapter.

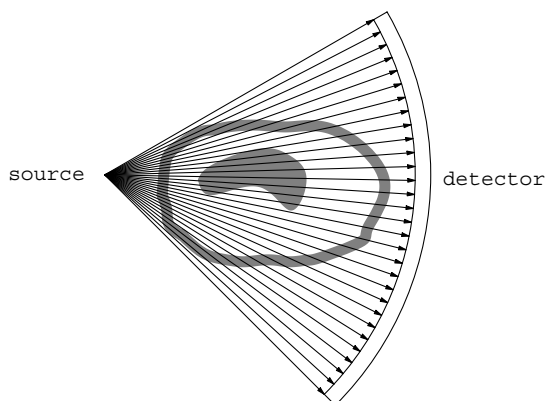
Some forms of tomography were used in diagnostic medicine long before computers were invented (see Section 1.10). A typical method attempted to visualize a section (slice) of a body by blurring out all the x-rays except those in the focal plane of the desired slice. Early CT scanners also concentrated on a single slice of the body. This attention on a slice (from *τομος*) explains the origin of the word tomography in medicine. The desire was to visualize a sliced human body without actually slicing it. In mathematical tomography the slicing refers to the lines or planes that slice through the object of interest.

Figure 1.2 is a picture of a typical CT scanner.

The circular ring in the CT scanner emits x-rays from a source on one side. These x-rays are detected at the opposite side. The ring rotates so that x-rays can be beamed, in any direction, through a specific slice of the patient's body. Here is a diagram of how this operates.

³ A CT scanner is also referred to as a CAT scanner, which is derived from “computer-assisted tomography,” whereas CT derives from “computerized tomography.” The preferred term is CT scanner, although CAT scanner is informally and ubiquitously used.

There is the story of the man who brought his sick dog to the veterinarian. Upon examination, the veterinarian pronounced the dog dead. The distraught owner replied: “That is impossible, I know my dog is listless, but certainly not dead. Is there not a more definitive test that you can do?” “Very well,” replied the veterinarian, who immediately summoned a black-and-white cat. The cat proceeded to examine the dog. First, the cat only sniffed around the dog who exhibited no reaction. Then the cat hissed at the dog and finally clawed it, all without reaction from the dog. The owner finally said, “I suppose you are right, my dog is dead. How much do I owe you?” The veterinarian replied, “That will be \$300.” The owner retorted, “Three hundred dollars to tell me my dog is dead! That is outrageous! Why is it so much?” “The veterinarian replied, “It is \$100 for the examination and \$200 for the CAT scan.”



This mode of scanning is called *fan beam geometry* for obvious reasons.⁴ CT scanners that use fan beam geometry are called *fan beam scanners*. In this mode x-rays are generated at the source. They form a beam in the shape of a fan and are observed at the detector after passing through the body. In this way the total density along every line emanating from the source can be computed. By rotating the apparatus, the source and detectors move to new positions. In this way the total density along every line intersecting the body can be determined. These total densities are the input data to an algorithm that reconstructs a picture of the organs and tissue in this slice. Later we will show how these total densities can be used to reconstruct an image of the tissue. Meanwhile, see figure 1.3 for a comparison of a traditional x-ray of the head and a tomogram of a section of a human brain. Note the lack of detail of the



Figure 1.3. (Left) traditional x-ray. (Right) Tomographic reconstruction of a brain section. Image on the left courtesy of Ass. Prof. Dr. Mircea-Constantin Sora, MD, Ph. D., Medical University of Vienna.

⁴ The analogous situation in three dimensions is called *cone beam geometry*.

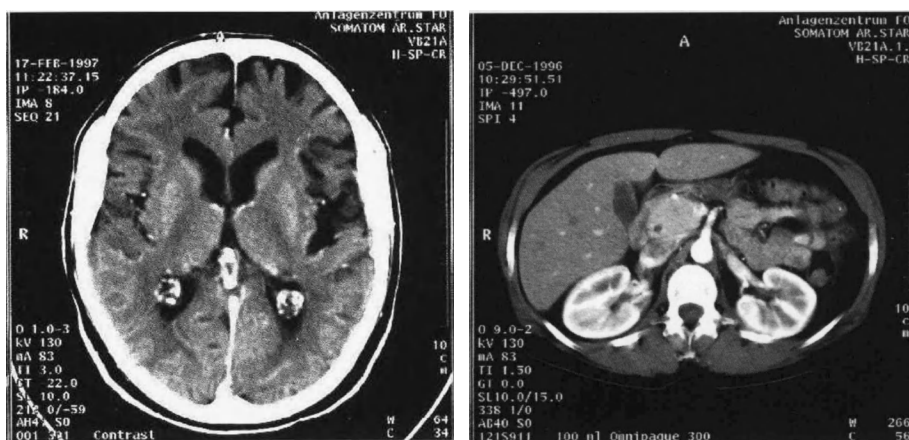
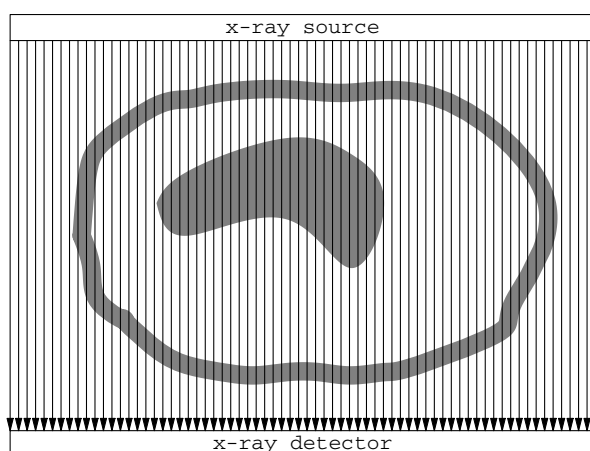


Figure 1.4. CT images. The left image is a cross section of a human brain. The right image is a cross section of a human abdomen. Courtesy of the Siemens corporation.

brain in the traditional x-ray compared with the fine detail in the tomogram. Another set of tomograms is in figure 1.4.

When an x-ray beam is sent through tissue, it experiences more attenuation by heavier tissue than by lighter tissue. For example, the skull is about twice as dense as the gray matter of the brain. Therefore, x-rays are more likely to be absorbed or scattered when passing through the skull than when passing through gray matter. Although there are some subtleties with this idea, we can make a working assumption that sending an x-ray beam through an object determines the total density of the object on the line intersected by the x-ray.

Before continuing we should mention that older CT machines used a parallel beam geometry. Their mode of operation is illustrated by the following figure.



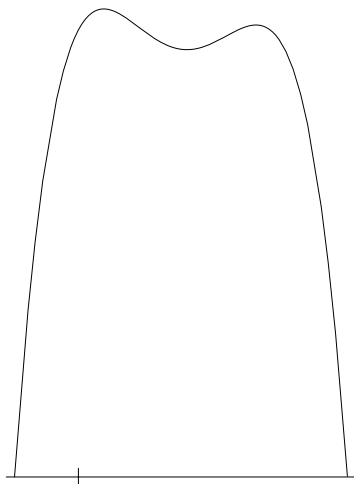
It is much more efficient to use fan beam geometry and most modern CT scanners use this method. In either method we can obtain information about the density of the object along any line, provided that the scanner is free to rotate through 180° . It is

simpler to describe the mathematics for parallel beam scanners and from now on we will do so. The method of reconstructing images via parallel beam geometry can be applied to fan beam scanners because, as we already noted, information about any line intersecting the object can be obtained in either geometry. However, the computational effort in reorganizing the fan beam data into parallel beam data is substantial. There are algorithms that use the fan beam data directly and these will be described in a later chapter. The efficiency of fan beam algorithms together with the efficiency of the fan beam scanning geometry make current CT scanners much faster than older ones. The latest generation of CT scanners emit x-rays along a helical path and reconstruct three-dimensional pictures.

1.3 Objects and Functions

Tomography is an example of a classical mathematical problem: determine an unknown quantity when some given information is provided. The unknown quantity might be a real number x , which is in some relation to some known real numbers, for example, $5x + 1 = 7$. In other situations the unknown quantity might be a function with some given information about its behavior. For example, determine the unknown position of a particle given its acceleration, its initial position, and its initial velocity. In a tomographic problem the given information is a set of x-ray projections of an unknown object. The solution is an exact or approximate representation of the unknown object obtained by mathematical manipulation of the known x-ray data.

At this point we need a precise definition of the term “object.” Let us take a simple object, say a two-dimensional image of the profile of a mountain:



To specify this object mathematically, all we need to know is the height of each point of the curve above the ground.⁵ Such a specification is called a function. Many

⁵ In this example the exact height is given by $-\frac{1}{4}x^4 - \frac{3}{8}x^3 + \frac{1}{2}x^2 + \frac{3}{8}x + \frac{27}{4}$, where the unit for x is denoted by the mark on the ground line in the picture.

two-dimensional shapes can be represented by functions in this way. In general, a **function** is a rule that uniquely assigns a value to each element of a given set. The given set is called the **domain** of the function. In this chapter we assume that the value assigned to an element of the domain is a real number.

An example of a function is given by the rule f which assigns $\frac{1}{x}$ to every nonzero real number x . Here the domain is the set of non-zero real numbers. We can define this rule by writing the equation

$$f(x) = \frac{1}{x}$$

Note that we use a letter, in this case f , to represent the rule or function. Then, for each x in the domain, $f(x)$ represents the quantity obtained by applying this rule to x . The symbol $f(x)$ is read as “the value of f at x ” or, in brief, “ f of x .” It is important to conceive of a function as a single object and not to confuse f with $f(x)$. However, sometimes we are sloppy and use $f(x)$ to denote the function f , even though $f(x)$ actually is a real number representing the *value* of the function f at x .

It is not necessary for the domain of a function to consist of numbers. For example, we can consider the function h , which assigns to every horse its weight in kilograms. In this case the domain is the set of all horses.

In the profile of the mountain, the domain was one-dimensional (the ground line). The function f contained all the information needed to describe the mountain’s profile. Although the profile of the mountain is two-dimensional, the amount of information needed to determine the profile is only one-dimensional. This is because the function describing the elevation is of the form $f(x)$, where x is a single variable that moves along a straight line.

In general, a two-dimensional object will require two variables to be completely determined. A general point in the plane is uniquely determined by the coordinates (x, y) . Therefore, we can treat more general two-dimensional objects by specifying a function of two variables: $f = f(x, y)$.

Figure 1.5 is an image of an abdominal section of a human patient. The density of the tissue at each point is depicted by the amount of gray at that point. The black points are the most “gray” and represent zero density. The actual tissue has varying density. The highest-density points are the least “gray” (white) and denote bone. The other tissue is less dense and is represented by various shades of gray. To describe this picture, all we need to know is the relative amount of gray to put at each point. This amount can be specified by a real number. Therefore, for all practical purposes, this image can be represented by a function f of two variables: $f(x, y)$ represents the amount of gray to place at the point (x, y) to create this picture.

An object can thus be viewed as a function of two real variables x and y , because we are uniquely assigning a real number (a gray value) to every point (x, y) of the plane. Conversely, if we are given a function of two variables, then we can view the associated object by using the value of the function at every point as a gray value. So from the mathematical point of view, there is no difference between a function of two variables and a two-dimensional object. For this reason we use the term “object” and

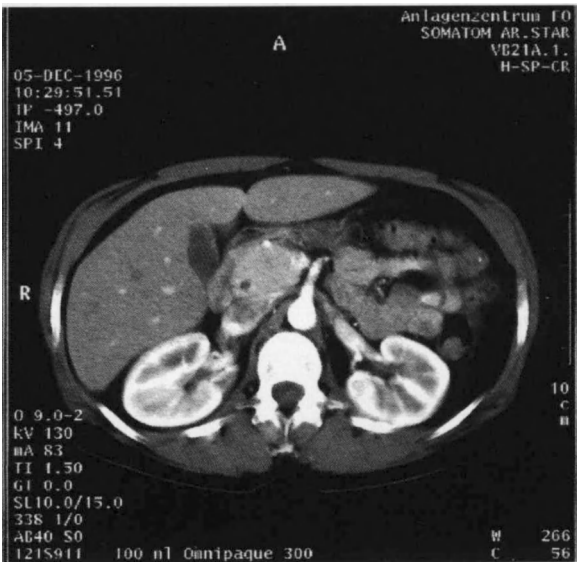
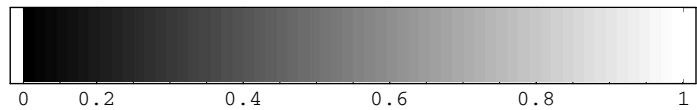


Figure 1.5. Tomogram of a human abdominal section. Courtesy of the Siemens Corporation.

the term “function” interchangeably. Also, for this reason we use letters like f and g to represent objects.

There are two main ways to exhibit the value of a function at a point. The first way is called a **density plot** and it attaches a gray level to each point in the domain. An example of a density plot is the tomogram in figure 1.5. Each gray level represents a specific real number, the lowest values of the function shown in black and the highest values shown in white.

The gray scale presented in the next figure shows how any real number between 0 and 1 can be represented as a gray level. It is not a function, it only serves to establish the correspondence of gray levels to a range of real numbers. The gray scale plays the same role as the x axis in a graph – it shows how we represent real numbers. One purpose of the gray scale is to establish the range of numbers used in the graph of the object. The range does not have to be from 0 to 1. It could be from any real number a to any larger real number b . However, the smallest value will always be represented as black and the largest as white. After this example, we will not be fussy about the actual range of values, so we will present objects without the accompanying gray scale.



The other way of graphing an object is to place a point of height $f(x, y)$ above the location (x, y) in the plane. This type of view is called the **graph** (of the object). An example of a graph of an object may be found in figure 1.6.

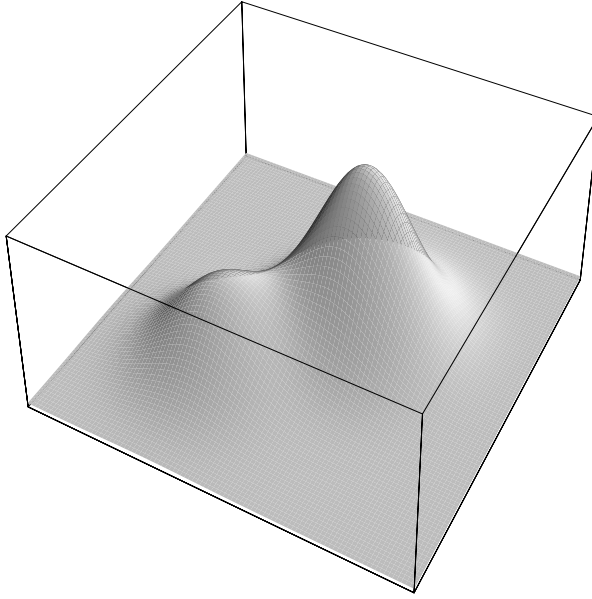
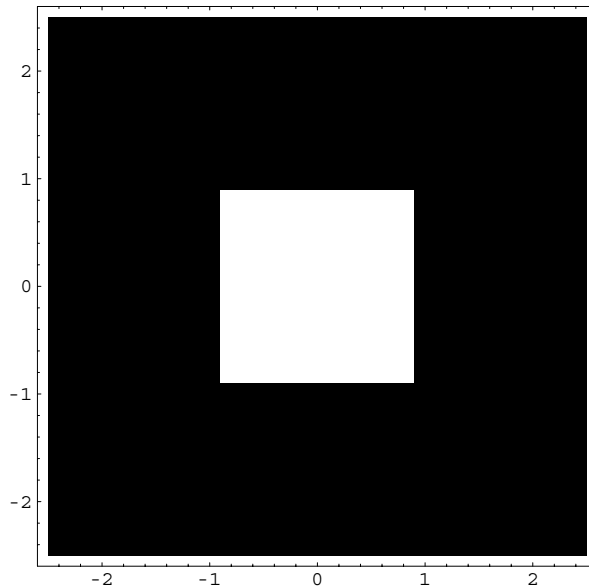


Figure 1.6. Graph of a function representing a mountain.

Simple objects may be represented by functions that take the value 1, represented by white, at all points that lie on the object, and that take the value zero, represented by black, elsewhere. Therefore, their density graphs will be exactly the shape of the object. To avoid becoming overly wordy let us agree that when we use a term such as triangular object we really mean the function that is 1 on the triangle and zero elsewhere.

Here is the density plot of a square object.



Here is its graph.

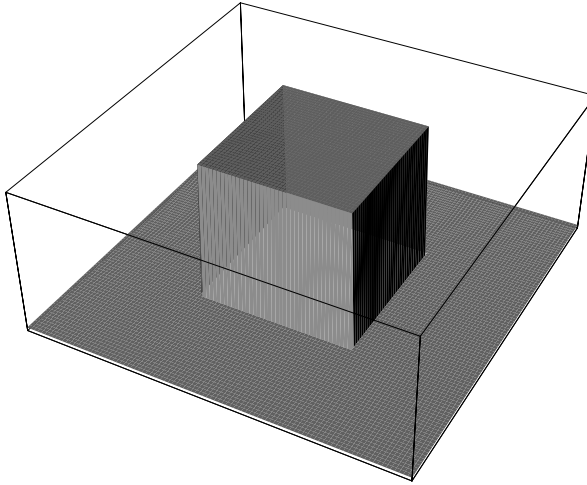
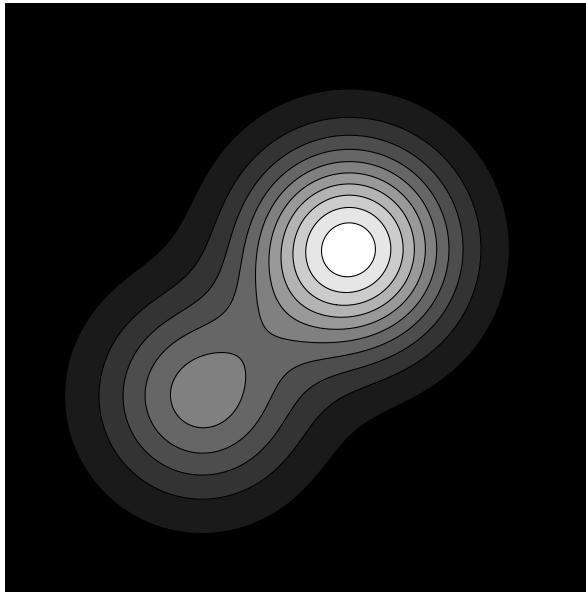
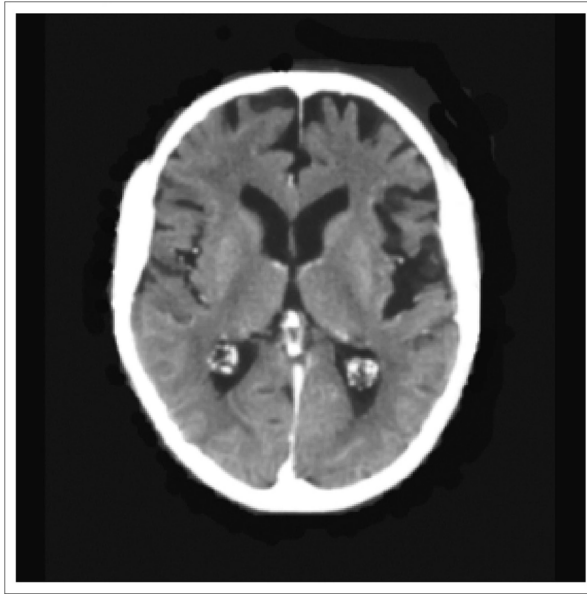


Figure 1.6 is a graph of a three-dimensional mountain. This is the three-dimensional analogue of how we represented a mountain earlier. In that case we placed the value $f(x)$ above the location x on a line. Here we place the height $f(x, y)$ above each point (x, y) in the plane.

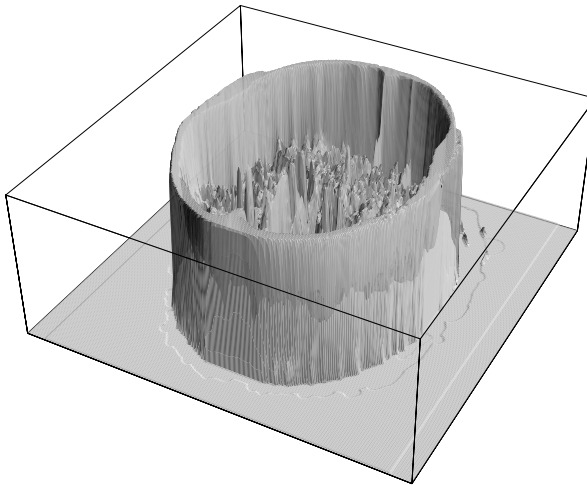
Note that the graph of the mountain appears three-dimensional (as it should). But it actually represents a function whose domain is two-dimensional. Next we show the density plot of the mountain. It is similar to contour plots of mountainous areas and is plotted on a two-dimensional domain.



Next we show a density plot representing a transverse slice of a human cranium.



The next figure is a graph of the same cranial section. It appears three-dimensional, but because it represents a transverse section of a cranium, it actually corresponds to a two-dimensional object.



It is clear from these images that for some objects the density plot is a better representation than the graph. This is true for the square object and the brain image. On the other hand other objects may be better represented as graphs, as for the mountain. Most of the objects we need to consider will be represented as density plots. However, the graphical representation is also useful and will be used if needed.

1.4 Tomography and the Radon Transform

We mentioned earlier that an x-ray beam can determine the total density of an object along the line of travel of the x-ray. We will investigate this process in more detail later (see section 1.9). For now we make the working assumption that one can determine the total density, or amount of material, along any line intersecting a specific object. Recall that the total density of an object along a line is called an x-ray projection of the object along that line. It is immaterial whether the knowledge of these x-ray projections was obtained via x-rays or by some other method.

We now examine the situation when the densities are known for all possible lines. The term **Radon transform** (of an object) is used to describe the collection of x-ray projections of an object along all possible lines.⁶ The problem of determining an object from its x-ray projections can therefore be formulated in the following way. Given the Radon transform of an unknown object, find the object. This is a typical problem in tomography. Before solving this problem, let us investigate the Radon transform in more detail.

The Radon transform of an object or function⁷ consists of x-ray projections along all possible lines in the plane. Each line has a specific direction and each direction is uniquely identified by an angle. Conversely, given an angle, we can specify a unique direction. Therefore we can use the terms angle and direction interchangeably. The symbol θ is often used to denote an angle; therefore, we will talk about directions θ . We specify directions in radians that we usually express in multiples of π . Recall that a full circle can be measured as an angle of 2π radians, which is the same as 360 deg. Therefore, $\frac{\pi}{4}$ represents a 45° angle because $\frac{\pi}{4}$ is exactly one eighth of a full circle measured in radians, whereas 45° is exactly one eighth of a full circle measured in degrees. With this in mind, the reader should have no trouble visualizing any angle or direction.

We now show that we can think of the Radon transform of an object as a function of two variables. Although there are many ways of doing this, we choose the method based on the following conceptual diagram of a CT scanner (fig. 1.7): In this diagram θ denotes the direction made by the motion of the source-detector array. This direction is indicated by the arrow perpendicular to the x-ray beam. If we specify a direction θ and a distance s from the origin, then there is exactly one line in the plane that is both perpendicular to θ and at the specified distance s . In figure 1.8, the thick solid line is perpendicular to the direction $\theta = \frac{\pi}{4}$ and at the distance $s = \frac{1}{2}$ from the origin of the coordinate system. The coordinate axes are dashed and an arrow indicates the direction of the angle θ . We intend to take the x-ray projection along this thick line.

If we move the thick solid line parallel to itself, then its distance s from the origin changes, but the line remains perpendicular to the direction θ . In this way we can obtain

⁶ In mathematics, when a function is defined with a domain consisting of other functions, then the new function is called a **transform**. Because the Radon transform assigns a set of x-ray projections to an arbitrary object (i.e., function), it is a function operating on other functions, so it is considered to be a transform.

⁷ Recall that, mathematically speaking, “object” = “function.”

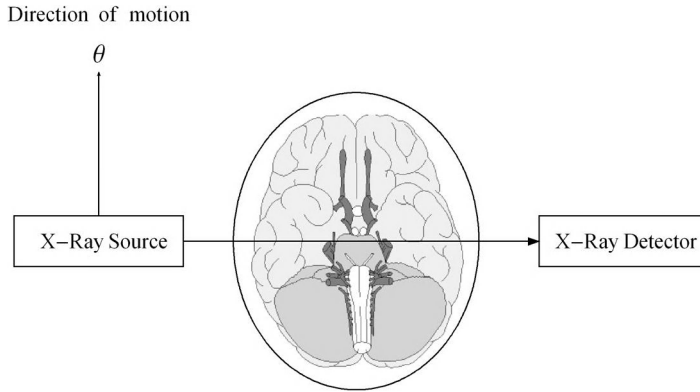


Figure 1.7.

the x-ray projections on all lines perpendicular to θ . Then by letting the direction θ vary over a range of 180° we can obtain the x-ray projections of an object over all possible lines in the plane.⁸ Hence, we will have the Radon transform of the object.

It may seem unusual to let the angle θ represent the direction perpendicular to the direction of the x-rays, but we see that this is exactly the mechanism in figure 1.7, which is typical of early CT scanners.

This shows that we can view the Radon transform of an object f as a function Rf of two variables (θ, s) : the symbol $Rf(\theta, s)$ denotes the x-ray projection of the object f along the line that is perpendicular to θ and which is located s units from the origin.

More informally, if we know the Radon transform of an object f , then we have a way of knowing the density of f along any line. Later on we will provide a graphic way of describing the Radon transform.

The notation $Rf(\theta, s)$ is most useful when we want to consider all the possible x-rays. It is useful to introduce a variation of this notation when we are dealing only with the x-rays in a single direction. For a given, fixed, direction θ , define

$$R_\theta f(s) = Rf(\theta, s)$$

This is the function of one variable which takes s into $Rf(\theta, s)$ which is precisely the **x-ray projection of f in the direction orthogonal⁹ to θ** . Therefore, $R_\theta f(s)$ represents the total density of f along the line which is perpendicular to θ and which is located s units from the origin.

For example, consider the line that is perpendicular to $\theta = \frac{\pi}{4}$ and located a distance of $s = \frac{1}{2}$ units from the origin, as in figure 1.8. Supposing that the total density of an

⁸ We also allow s to be negative. In this case we interpret the distance s from the origin to be in the opposite direction of θ . In this way varying θ over 180° gives all the lines in the plane.

⁹ It is common to use the term “orthogonal” as a synonym for “perpendicular.”

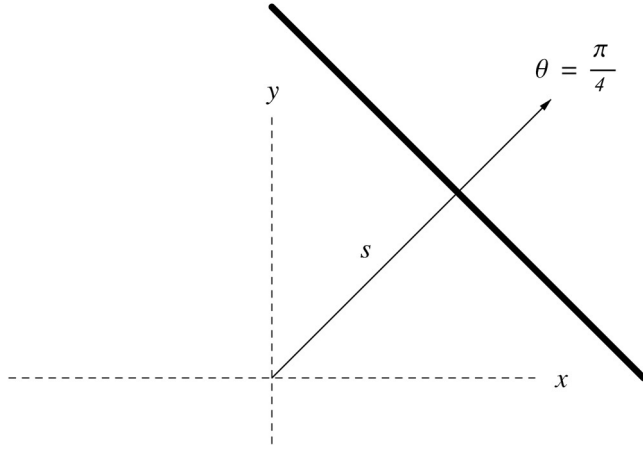


Figure 1.8. Geometry of x-ray projections.

object f along this line were 3.5, then we would write

$$R_{\frac{\pi}{4}} f \left(\frac{1}{2} \right) = 3.5$$

This is exactly the same quantity as $Rf(\frac{\pi}{4}, \frac{1}{2})$ but in a different notation.

Now we take a simple object, a square, and see how to compute its Radon transform. We begin by computing a single x-ray projection. Let f be the function representing the object that is a square of side 2 centered at the origin. Recall the earlier discussion in which we represented simple objects by functions that take the value 1, represented by white, at all points that lie on the object and that take the value 0, represented by black, elsewhere. Figure 1.9 shows a density plot of this square object. Also shown is the line perpendicular to $\theta = \frac{\pi}{4}$ at a distance $s = \frac{1}{2}$ from the origin. The value of the x-ray projection of f can then be easily calculated. Because black areas represent 0 and white areas represent 1, we can compute $Rf(\theta, s)$ by measuring the length of the intersection of the line with the white square. This is because the total density at each black point is 0 and at each white point is 1.

The equation of the given line perpendicular to θ and distance $s = \frac{1}{2}$ from the origin is $y = \frac{\sqrt{2}}{2} - x$. This line enters the square at the point $(1, \frac{\sqrt{2}}{2} - 1)$ and exits the square at the point $(\frac{\sqrt{2}}{2} - 1, 1)$. The x-ray projection of the square along this line is thus the distance between these points,¹⁰ so $Rf(\frac{\pi}{4}, \frac{1}{2}) = 2\sqrt{2} - 1 \approx 1.8$. It would be more difficult to compute the x-ray projection of objects with varying density, but

¹⁰ Recall the distance formula between points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$:

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Applying this formula to $P = (1, \frac{\sqrt{2}}{2} - 1)$ and $Q = (\frac{\sqrt{2}}{2} - 1, 1)$ gives the resulting distance. $\sqrt{9 - 4\sqrt{2}}$ which simplifies to $2\sqrt{2} - 1$.

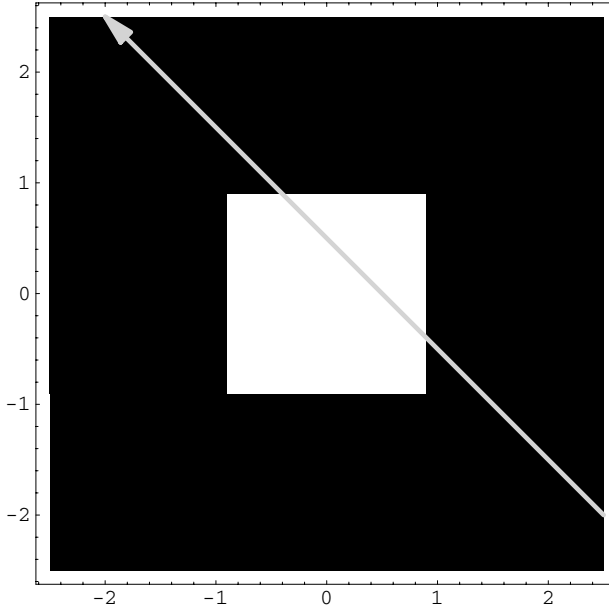


Figure 1.9. A single x-ray projection.

the basic principle is the same. In CT this computation is automatically derived from the x-ray data.

Using the x-ray projection notation, $R_\theta f(s)$, for $\theta = \frac{\pi}{4}$ and $s = \frac{1}{2}$, we have

$$R_\theta f(s) = R_{\frac{\pi}{4}} f\left(\frac{1}{2}\right) = 2\sqrt{2} - 1$$

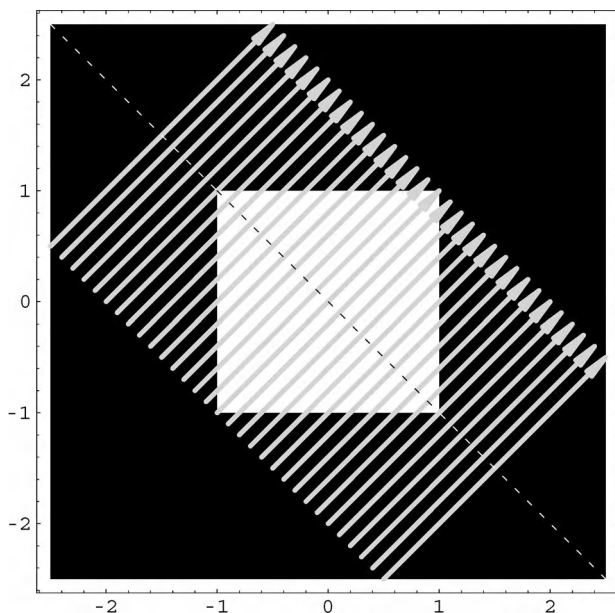
Other x-ray projections can be computed in a similar way. The easiest example is for $\theta = 0$, because all lines perpendicular to $\theta = 0$ are vertical. The x-rays are therefore all vertical and they either intersect the square with the identical length 2 or else they completely miss the square. Therefore,

$$R_0(s) = 2$$

for any s between -1 and 1 , whereas $R_0(s) = 0$ for any other real number s .

Figure 1.10 illustrates a sample of the full x-ray projection, this time in the direction orthogonal to $\frac{3\pi}{4}$. Using the notation for projections we can then say that this figure represents the x-ray projection $R_{\frac{3\pi}{4}} f$ where f is the square object.

All the x-rays in figure 1.10 are in the 45° direction, because they are perpendicular to $\theta = \frac{3\pi}{4} = 135^\circ$. When we look at this figure we see that we have $R_{\frac{3\pi}{4}} f(s) = 0$ for very negative values of s (those less than $-\sqrt{2}$, represented by the lines in the black, upper left area of fig. 1.10, recall that s represents the signed distance of the x-ray from the origin). This is because in this area of the figure, f has the value 0 and hence the

Figure 1.10. A complete projection, $R_\theta f$ orthogonal to $\theta = \frac{3\pi}{4}$.

total density of f along any of the x-rays in this area is also 0. Once $s = -\sqrt{2}$, the x-rays begin to intersect the white square. At first there is not much contribution to the x-ray projections, because for values of s greater than but close to $-\sqrt{2}$ the x-rays do not traverse very much of the white square. But once $s > -\sqrt{2}$, the values of $R_{\frac{3\pi}{4}} f(s)$ start to increase until they reach a maximum, when $s = 0$ (this will give a diagonal of the square). Then the values decrease in a symmetric way until $R_{\frac{3\pi}{4}} f(s)$ again becomes zero at $s = \sqrt{2}$. At this point the x-rays again do not intersect the square, so that the x-ray projections remain 0 for $s \geq \sqrt{2}$.

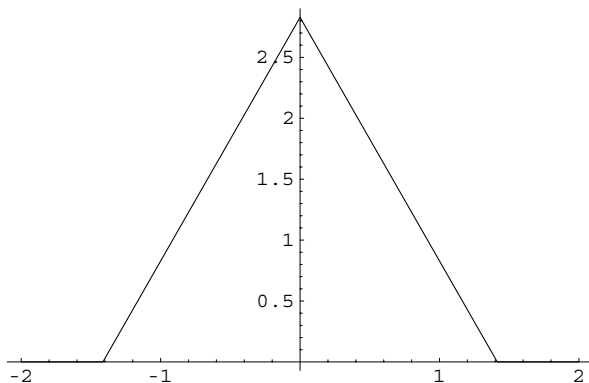
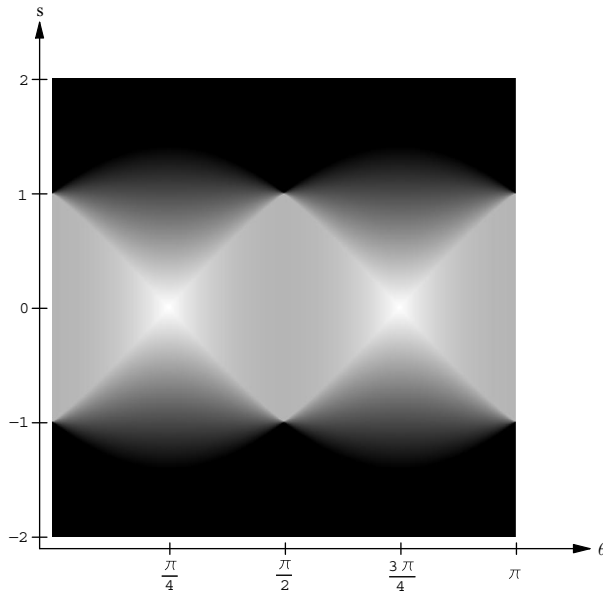
Figure 1.11. Graph of the x-ray projection $R_\theta f$ for $\theta = \frac{3\pi}{4}$.

Figure 1.11 shows the graph of $R_{\frac{3\pi}{4}} f$. This graph is plotted by computing the x-ray projection $R_{\frac{3\pi}{4}} f(s)$ for each real number s and then placing a point on the graph at a height of $R_{\frac{3\pi}{4}} f(s)$ units above the point s on the horizontal axis. You can see the increase in the values of $R_{\frac{3\pi}{4}} f(s)$ as s goes from $-\sqrt{2}$ to 0 as predicted. Also note the decrease as s goes from 0 to $-\sqrt{2}$.

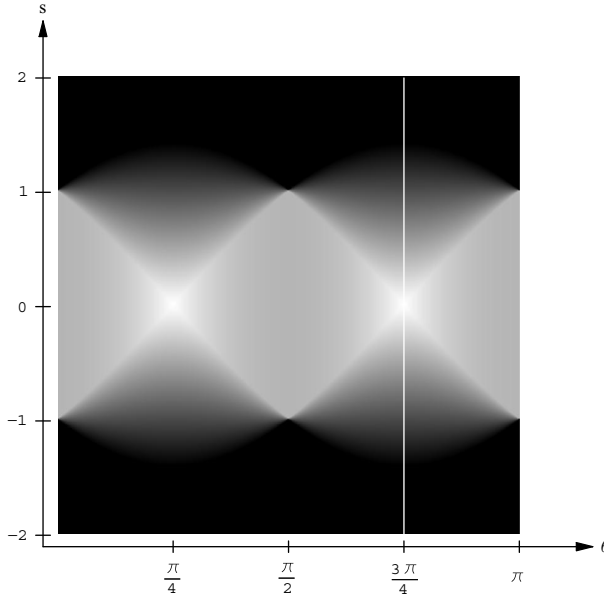
The next step in the study of the Radon transform is to devise a way of visualizing the entire Radon transform, instead of a single projection. The term *sinogram* is used for the density plot of the Radon transform.¹¹ Therefore we can create a sinogram by creating a density plot of the function of two variables $Rf(\theta, s)$.

Here is a figure showing the sinogram for the square object under consideration. The gray scale is set up so that black represents 0 and white represents $2\sqrt{2}$. The θ axis, which represents the directions, is horizontal and the s axis, which represents the distance from the origin, is vertical. To create this sinogram we merely have to place an appropriate gray value at the point located at the coordinates (θ, s) . This gray value corresponds numerically to $Rf(\theta, s)$.



Conversely, we reverse this process to read a sinogram. If we want to know the numerical value of $Rf(\theta, s)$, then we locate the point (θ, s) in the sinogram and we convert its gray value to a real number that is therefore equal to $Rf(\theta, s)$. This is illustrated in the next figure.

¹¹ Perhaps this unusual terminology has to do with the resemblance of a typical sinogram to sine waves. Maybe a better term would have been “Radonogram.”



Every point on the vertical white line through the point $\frac{3\pi}{4}$ on the θ axis is of the form $(\frac{3\pi}{4}, s)$ for values of s between -2 and 2 . The gray value at $(\frac{3\pi}{4}, s)$ represents $Rf(\frac{3\pi}{4}, s)$, or, equivalently, the x-ray projection $R_{\frac{3\pi}{4}}f(s)$. Between $s = -2$ and $s = 0$, the gray scale value increases from black to white and, hence, the x-ray projection $R_{\frac{3\pi}{4}}f(s)$ increases from 0 to $2\sqrt{2}$. Once we understand this, then we see that $R_{\frac{3\pi}{4}}f(s)$ decreases from 0 to $2\sqrt{2}$ when s varies from 0 to 2 . This coincides exactly with the graph of the x-ray projection seen in figure 1.11.

Therefore, we can obtain information about any x-ray projection $R_{\theta}f$ from the sinogram for f . Just erect a vertical line through θ and observe the gray values along this line. In figure 1.12 we again have the sinogram for the square object f . Also shown are graphs of x-ray projections in the three directions $\theta = \frac{\pi}{6}$, $\theta = \frac{\pi}{4}$, and $\theta = \frac{2\pi}{3}$. Note that the graph of each individual x-ray projection forms a function of one variable as the theory indicates. An arrow leads from the vertical line through $\theta = \frac{\pi}{6}$ to the graph of the x-ray projection $R_{\frac{\pi}{6}}f$. This graph makes sense, because the vertical line starts at $s = -2$, where the black area indicates a value of zero for the density. The first nonzero value occurs at about $s = -1.37$, where the graph starts getting progressively less black. Therefore, the values of $R_{\frac{\pi}{6}}f$ increase until a maximum is reached at about $s = -0.37$. Then the densities are constant until about $s = 0.37$ at which point they decrease to zero at about $s = 1.37$. This gives rise to the trapezoidal graph for $R_{\frac{\pi}{6}}f$. You can check the plausibility of the other two x-ray projection graphs in the same manner.

Figure 1.13 contains examples of two human sinograms. Both plots represent the Radon transform for a section of a human body. One of these gives x-rays for a human abdominal section and the other for a human brain section. But which is which? And

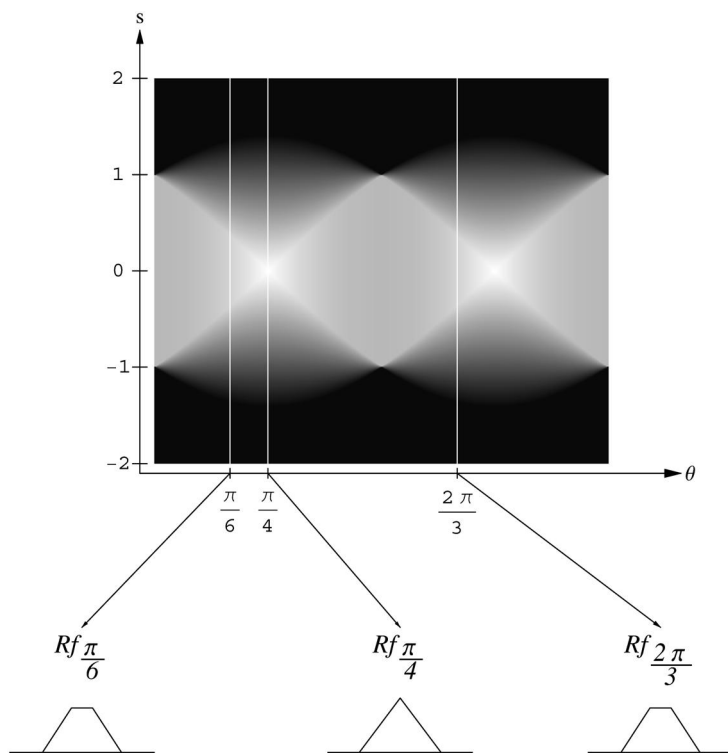


Figure 1.12. Sinogram generated by the Radon transform of the square object, with graphs of three x-ray projections.

what does each organ look like? It is hard to tell directly from the given information. We will determine the answer later.

So far we have seen that an x-ray projection, $R_\theta f$ orthogonal to a given, fixed direction θ , is a function of one variable representing the total densities of the object

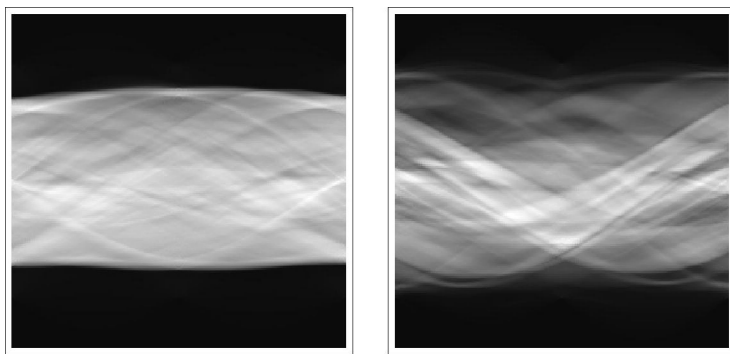


Figure 1.13. One of these graphs represents the x-rays through a human abdomen, the other represents the x-rays through a human brain. Which is which?

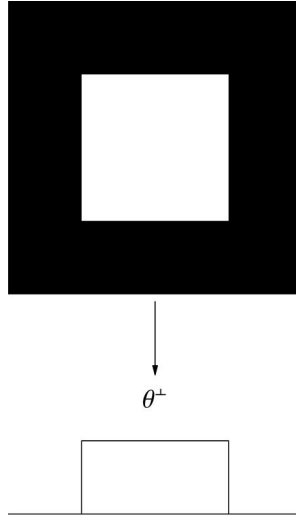


Figure 1.14. Single x-ray projection.

f in the given direction. The set of all such x-ray projections forms the Radon transform of the object. The problem in tomography is therefore to use the known x-ray data in the form of the Radon transform to reconstruct the unknown object from which it originates. The sinogram represents the known or given information for this problem.

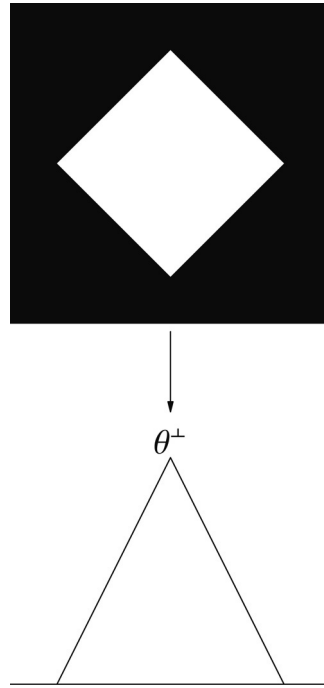
Soon we will give an explanation of how such a reconstruction can be accomplished. The explanation is based on the method used by CT scanners in medical applications. However, we first introduce an intuitively plausible method, which appears, at first glance, to yield a reasonable reconstruction from x-ray projections in a very simple manner.

1.5 Backprojection

The x-ray projection $R_\theta f$ of an object f is the total density of f along lines orthogonal to the direction θ . We now create a dual operation, called backprojection, which takes a certain amount of material and smears it backward along lines orthogonal to θ . Let us denote the direction orthogonal to θ by the symbol $\theta^\perp$.

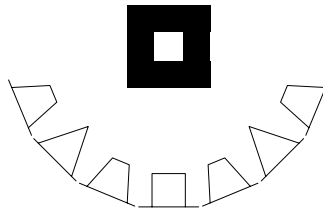
Let us think of the material in the object f as being made of sand. Orient the object so that gravity acts in the direction orthogonal to θ . If we allow the “sand” to spill out of the object and collect on a floor below the object, then this pile of “sand” will look like the x-ray projection $R_\theta f$ orthogonal to the direction θ .

Figure 1.14 illustrates this idea. In this figure we have taken $\theta = 0$. The material in the square object has been projected in the orthogonal direction $\theta^\perp$, thereby creating the pile of material shown at the bottom of the figure. If we choose a different direction, say $\theta = \frac{\pi}{4}$, then the situation looks like this (again we have oriented the object so that gravity acts in the direction $\theta^\perp$):



Here we get a triangular pile of “sand.” This can be verified by imagining the material in the object to pour “downward” (actually in the direction $\theta^\perp = -\frac{\pi}{4}$).

The next figure illustrates this idea for several directions. Each “pile” has been formed by projecting material in the white square parallel to the direction $\theta^\perp$ between the center of the square and the center of the “pile.” Each pile therefore represents the graph of an x-ray projection of the form $R_\theta f$.



We now describe the process of backprojection. The process is dual to the operation of taking projections. Instead of starting with an object in the plane and creating a pile of material underneath, we start with a pile of material and create an object in the plane. The object is created by smearing the material back into the plane and for this reason the process is called backprojection.

Let g be the function representing the pile of material. Figure 1.15 displays the result of backprojecting this function. The pile of material has been smeared, or

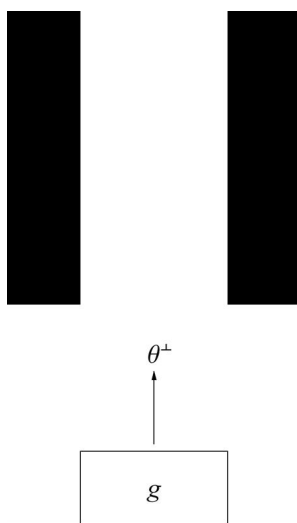
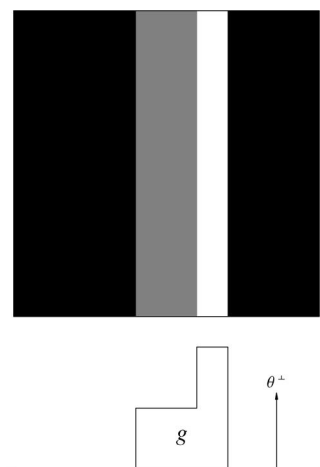


Figure 1.15. A single backprojection.

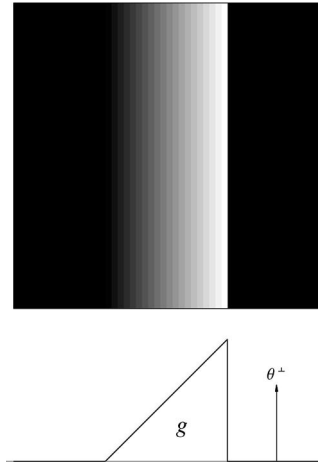
backprojected, throughout the entire plane in the direction $\theta^\perp$. The white stripe consists of the material from the original x-ray projection which has now been backprojected. Here we started with a blank plane and a pile of material underneath and we created an object, the white stripe, by the process of backprojection, in one direction. This is called **backprojection in one direction**. If the function representing the material available to be backprojected is denoted by the symbol g , then $R_\theta^\# g$ denotes its backprojection in the direction orthogonal to θ . The part of the symbol denoted by $R^\#$ is meant to remind us of a connection to the Radon transform, although the connection is not yet apparent.

Here are some more examples of this idea.



Note that the function g takes on three values: it is 1 between $-\frac{1}{2}$ and $\frac{1}{2}$, 2 between $\frac{1}{2}$ and 1, and 0 elsewhere. After backprojecting we therefore obtain four strips: two are black, corresponding to the zero values; one is gray, representing the value 1 for all points with t – coordinate between $-\frac{1}{2}$ and $\frac{1}{2}$; and one is white, corresponding to the value 2 for points with t – coordinate between $\frac{1}{2}$ and 1.

Here is one more example of a backprojection in one direction. In this case the function g takes on all values from 0 to 1:



In the backprojection you can see the variation from black to various shades of gray to white and then back to black corresponding to the change in the function g from 0 to 1 and back to 0.

We would now like to perform backprojection in several directions. It makes sense to average the contributions from all these backprojections, otherwise the reconstructed object would grow to be too large. The process of averaging several backprojections from different directions is simply called **backprojection**. We use the notation $g = g(\theta, t)$ for the functions involved in the backprojection process. If we fix a particular θ , then $g(\theta, t)$ is a function of one variable that plays the role of g in backprojection in a single direction. This device allows us to backproject in any direction.

Let us now backproject the Radon transform of a simple object: the square f of side 2 centered at the origin. We saw how to compute the Radon transform in Section 1.4 and we visualized the backprojection in a single direction in figure 1.15. Let us now start backprojecting in more directions.

The following diagram shows the effect of averaging the backprojections from three directions. Note that the functions being backprojected are now x-ray projections

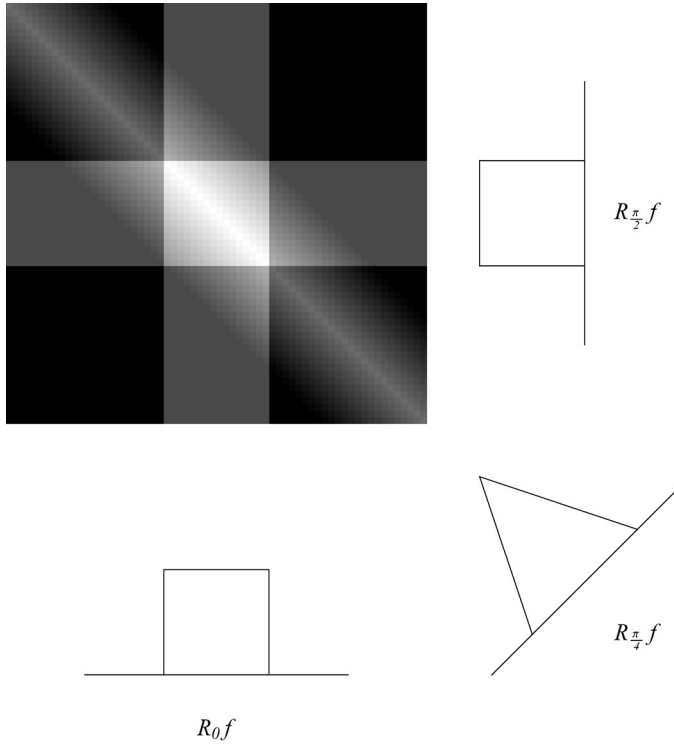
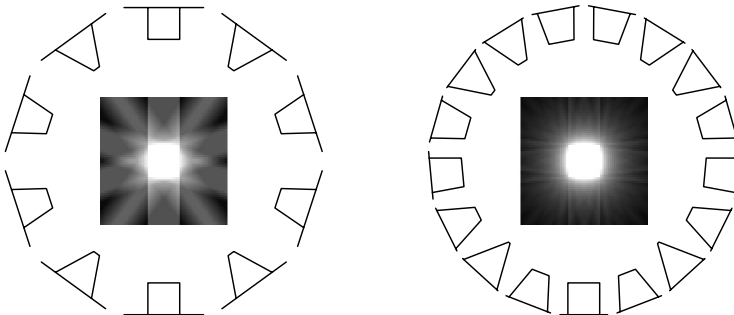


Figure 1.16. Three averaged backprojections of the Radon transform of the square object f . Recall that $R_\theta f$ is the notation for the x-ray projection of f along the line perpendicular to the direction θ . Here we have $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$.

orthogonal to angles $0, \frac{\pi}{4}$, and $\frac{\pi}{2}$, namely the projections $R_0 f$, $R_{\frac{\pi}{4}} f$, and $R_{\frac{\pi}{2}} f$, where f is the square object introduced before.

Here are the results of backprojecting in 10 and then in 17 directions.



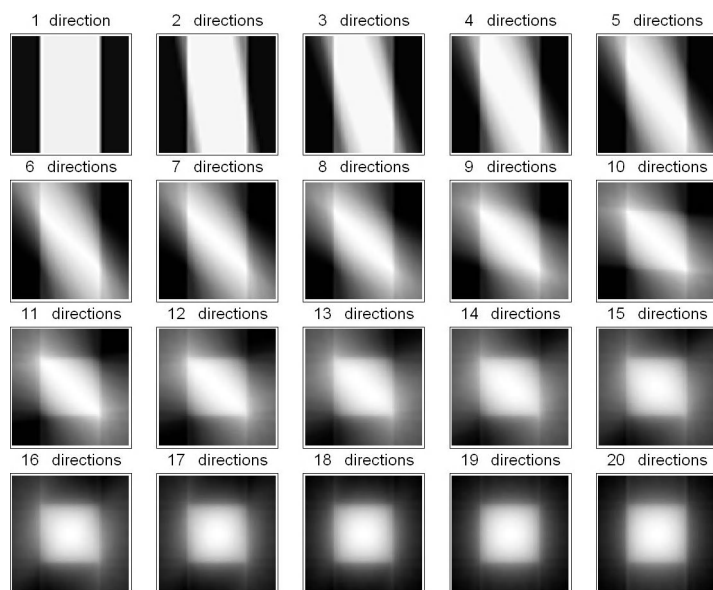


Figure 1.17. Successive backprojections of the Radon transform of a square.

It seems that the more directions used to back project, the more the result resembles the original square.

Figure 1.17 shows a backprojection formed by averaging 20 individual backprojections. The first picture in the upper left of the figure is precisely the single backprojection discussed previously (fig. 1.15). The second picture illustrates the average of this backprojection with one from a neighboring direction. The complete sequence illustrates successive averages of neighboring backprojections until a full range of directions has been achieved. Read the diagram from top left to bottom right, that is, in increasing order of the number of directions. The lower right image is the reconstruction of the square by backprojecting its Radon transform.

The image constructed by backprojecting and averaging through all directions is denoted by $R^\# g$, where $g = g(\theta, t)$ is the function described earlier. Recall that $g(\theta, t)$ represents the material piled up at the point t in a direction orthogonal to θ . Because all directions θ have been used to construct $R^\# g$ from g , a dependence on θ no longer exists. The result of backprojection, the function $R^\# g$, represents an object in the plane and hence is a function of the two variables x and y . It is amazing to see the reconstruction take shape from a very different initial picture through some formless blobs to finally more and more square shapes. It seems very encouraging that we have found a simple method of reconstructing a function from its Radon transform. Let us look at the results of backprojecting the Radon transform through 500 directions.



It appears that we may have reconstructed the square.

However, if we examine the graphs of the reconstruction and the original object, then we see that the reconstruction is not all that good. In Figure 1.18 we see the result of backprojecting the Radon transform of the square object f from 500 directions: We chose 500 directions because later on we will describe a modified backprojection process that gives very good reconstructions for simple objects using about this number of directions. However, the process described here gives very poor results even for simple objects. Even if we used more than 500 directions, there is no hope of doing much better from the backprojection process described here. This is true even if we use infinitely many directions. The process of taking the backprojections of the Radon transform of an object yields what is called a Riesz potential of the object. The Riesz potential is almost never very close to the object itself, so this method cannot give very good reconstructions. The more advanced reader can check this out in the discussion following the proof of theorem 2.75 in chapter 2.

The general shape of the reconstructed object is that of a square when viewed from above. But viewed as a three-dimensional graph the original object is a rectangular prism, whereas the reconstruction is far from that as figure 1.18 shows. There are ways

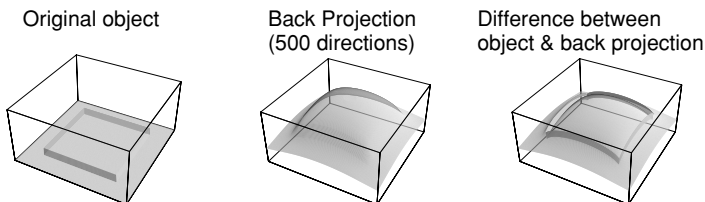


Figure 1.18. Reconstruction of a square by backprojection. (*Left*) Original object. (*Center*) Reconstruction by backprojection. (*Right*) The difference between the original object and its reconstruction is larger than the original object itself!

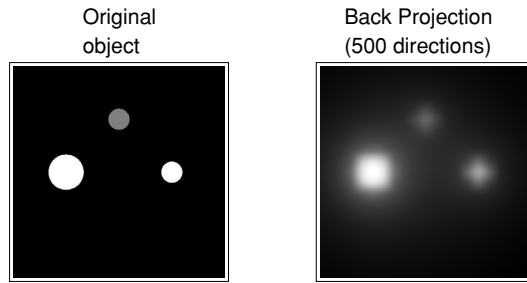
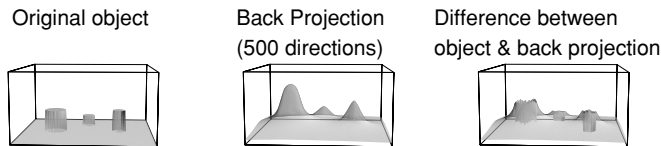


Figure 1.19.

of correcting for the poor reconstruction. One way is to look at the difference between the original object and its reconstruction. This difference can be considered to be a form of noise. Then noise reduction procedures may give a better reconstruction. We will describe this in the next section, but let us first look at another example.

This example (fig. 1.19) involves an object consisting of three disks (on the left) and its reconstruction by backprojection (on the right). Again, the reconstruction shows some of the features of the original object. But there are other features that are not part of the original picture. This is more apparent by examining the graphs:



These examples show the limitations of the backprojection process. Although the reconstructed images have some of the essential features of the original object, they also have numerous features that are in error. These other features are called **noise** or **artifacts**.

Therefore, it is necessary to try to obtain better reconstructions than can be obtained by the backprojection method.

1.6 Noise Reduction and Error Correction

To a mathematician or engineer, noise is simply the difference between an object and its reconstruction. An engineer deals with objects called signals. A transmitted signal is received (reconstructed) but perhaps contaminated with noise. A mathematician deals with objects represented by functions. The mathematician may try to reconstruct a given, but unknown object by backprojection. We saw that the process of backprojection created noise: a difference between the original object and its reconstruction. The idea now is to eliminate all or at least most of the noise.

In the transmission of signals, a way of eliminating noise is to apply an averaging process called **filtering**. We will investigate this process and try to adapt it to reconstructing functions by backprojection.

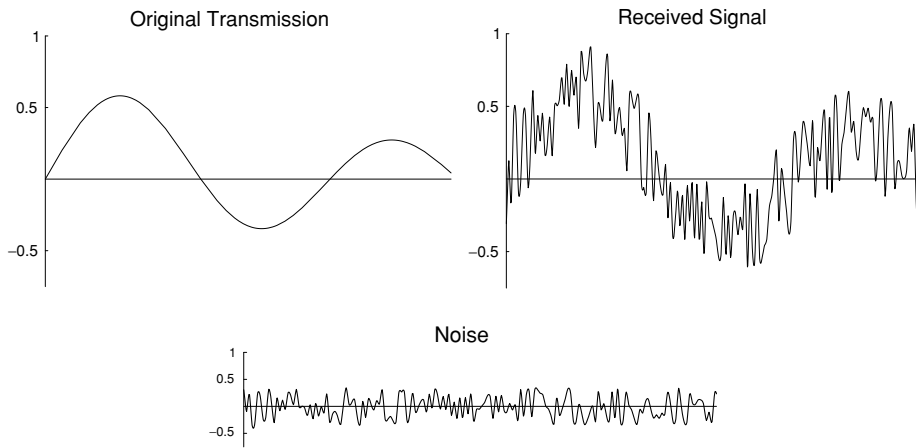


Figure 1.20.

To introduce the filtering process, we consider a signal represented by a graph of a single real variable. This signal is then transmitted over a noisy medium and received as a corrupted version of the original transmission. Figure 1.20 shows a transmitted signal and its corresponding received signal. The received signal is corrupted by high-frequency noise. The noise is the difference between the original transmission and the received signal. This type of signal occurs in radio or voice transmission. In radio transmissions, the noise could be caused by lightning or other atmospheric disturbances. For other types of transmission, such as sound, ultrasound, x-rays, etc., noise could result from any disturbance to the process of transmission or to the medium through which the signal is transmitted.

Notice that the noise in the transmission is also shown. As mentioned, this noise is merely the difference between the original signal and the received signal. Observe that the noise has almost identical positive and negative contributions. Thus if one averaged the noise, then the noise would mostly disappear. There are various ways of averaging and, in mathematics, these are performed by a process called **convolution**. A convolution takes a signal (i.e., a function of one variable) and averages it with some fixed function called a **filter**. The filter characterizes the type of averaging desired. You can understand the idea of filtering and convolution by drawing a noisy graph. Then take an eraser to try to blur the graph into a smoother shape. The graph represents the signal, the eraser represents the filter, and the process of applying the eraser to the graph is what we mean by convolution. In more precise terms, convolution is a process of moving averages: each portion of the graph is smoothed out by averaging the values in a neighborhood of each point. This process is also called **filtering** the signal. A different choice of eraser, say an eraser with a different shape or thickness, would correspond to a different convolution. The mathematical notation for a convolution is $f * k$, where f is the function being processed (signal) and k is the filter (eraser). The notation $f * k$ is read “the convolution of f and k .” In chapter 2, section 3 we will see that the filter can be represented by a function and that the convolution $f * k$ is an integral of some sort involving these functions. If you have not studied the calculus, it

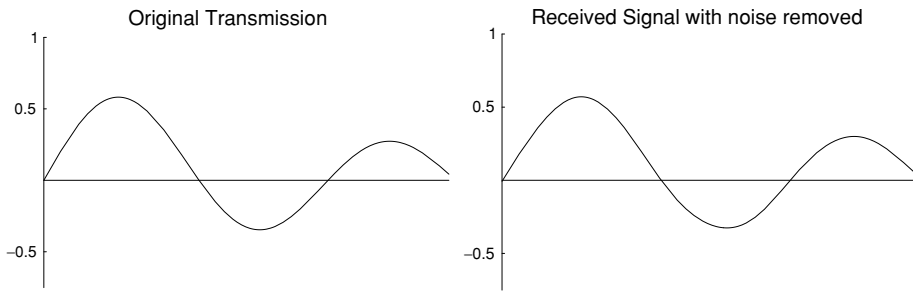
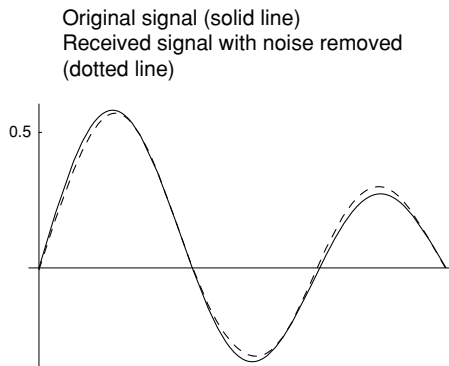


Figure 1.21.

is enough to understand that the process of forming an integral is a summation process that can be adapted to computing averages.

Figure 1.21 shows both the original transmission and the received transmission, after the convolution process has removed most of the noise.

Here are some more comparisons between the original signal, the noisy received signal, and the reconstructed signal with noise removed:



You can see that the reconstructed signal is close to but not identical with the original signal. It is, however, a much better approximation to the original transmission, as you can see in the next figure in which the noisy received signal is superimposed on the original signal.

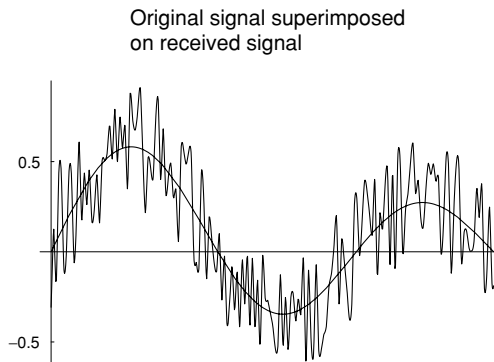


Image contaminated with noise

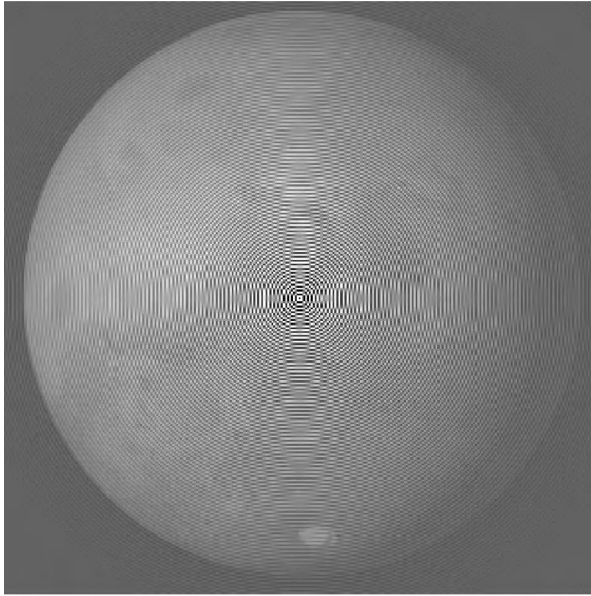


Figure 1.22. Image corrupted by high-frequency noise.

The idea of filtering out noise also applies to image transmission. In this case a function of two variables is being transmitted. The process of transmission frequently collects noise and the received image is garbled. This actually happens in space exploration where a space vehicle takes a high-resolution picture of a planet then transmits the image back to Earth. The transmitter is typically weak, to save weight on the space craft. Therefore, the transmitted signal may be corrupted by noise caused by the weak signal in conjunction with various disturbances encountered on the journey through space. Figure 1.22 is a simulated image that might have been sent by a space craft.¹² After filtering the image looks like fig 1.23. You may recognize it as a view of the planet Mars. The original image appears in fig. 1.24. Comparing this to the filtered image, we see that the filtering process has eliminated most of the noise.

In the two examples of noisy transmissions already considered, the noise had a random, high-frequency nature. We have seen that applying a certain type of filter to the noisy signals can eliminate the noise. We also see that there is a penalty: the filtered image is somewhat fuzzier than the original image; it is only an approximation to the original not an exact reconstruction. However, for most purposes, including diagnostic medicine, we can obtain approximate reconstructions good enough to distinguish diseased tissue and pathologies from healthy tissues.

The filtering process for noise reduction is so elegantly successful that we should investigate whether the idea can be adapted to image reconstruction from projections.

¹² For this example, the noise has been added by a computer program, but the idea is the same as in real transmissions.

Contaminated image after noise removal

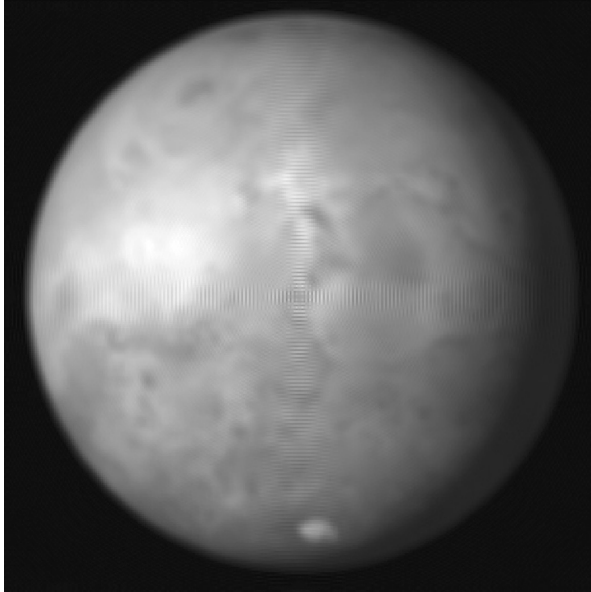


Figure 1.23. Noisy image after filtering. Notice the reduction in noise.

Original Image

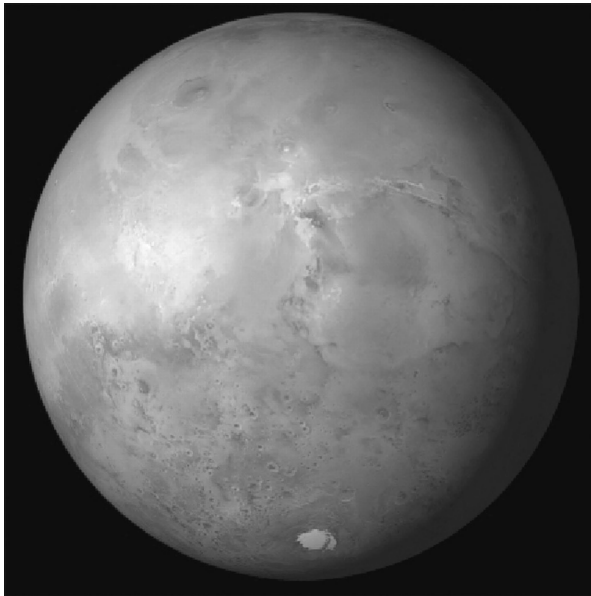


Figure 1.24. Original image (courtesy of NASA.)

Indeed this can be done, but as with the filtration of high-frequency noise, the best we can expect is a good approximation to the original object.

1.7 Filtered Backprojection: How a CT Scanner Reconstructs a Picture

Filtered backprojection is the same backprojection process described before except that each x-ray projection is filtered before it is averaged into the backprojection. So you should think of the filtered backprojection process as an amalgamation of the backprojection idea with the noise reduction process. The mathematical details are presented in chapters 2 and 3, so for now we only present graphical images illustrating the technique.

The main difference between filtered backprojection in tomography and noise reduction in image processing is that in tomography there is no noise in the known data, the x-ray projections,¹³ whereas, in image processing there is noise in the known data, the received image. The noise in tomography comes from the backprojection process itself. However, just as in image processing, a filter is applied to the known data, the x-ray projections. Then, when these are backprojected, the noise is mostly eliminated. The resulting picture will be a good reconstruction of the original object, if the filtering is done in the correct manner.

The reason why this process works will be covered in chapter 2, section 7. For now we present two examples that graphically illustrate the process. We show the progression of filtered backprojections that reconstructs the unknown function, but we do not describe the filter of the convolutions that produce these results.

It is important to realize that any process involving filtering can only give an approximate solution to the problem. The examples given here, however, yield very good approximations.

The first example is a simple object. We do not know what it looks like at the start, but we do know its Radon transform. At the end of the reconstruction process we will see the original object to which we will compare the reconstruction.

The second example is a reconstruction of a human brain section from its x-ray projections.

1.7.1 Filtered Backprojection Reconstruction of a Simple Object

We are presented with an unknown object but are given its Radon transform. Let us call the unknown object f . The known data, the sinogram, is in fig. 1.25. It represents the information collected by sending x-rays through the object from all directions. As we know, this information is the Radon transform of f , and this graph is called the sinogram.

The unknown in this problem is the original object and the aim is to solve the problem by giving a good approximate reconstruction of the unknown f via filtered backprojection. Before doing this let us see what predictions we can make about the shape of the object directly from the x-ray data.

¹³ We are considering the ideal situation in which the actual x-ray projections are the known data. In practice noise also occurs in the x-ray projections and this issue must also be dealt with.

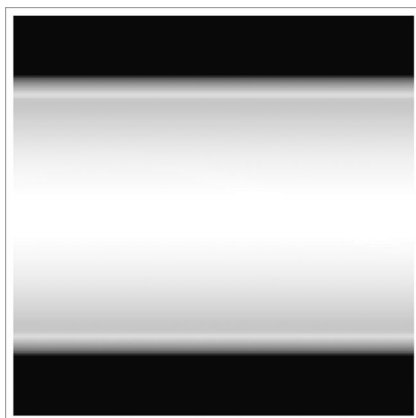


Figure 1.25. Sinogram of $Rf(\theta, s)$, the graphical representation of the Radon transform (x-ray data) of the unknown object f .

Recall that the horizontal axis represents the directions of the x-rays. If you erect a vertical line at any particular direction, the gray levels along that line determine the x-ray data in that direction. To the unaided eye it appears that these data are the same for every direction, so the object probably has circular symmetry. Also you can see that

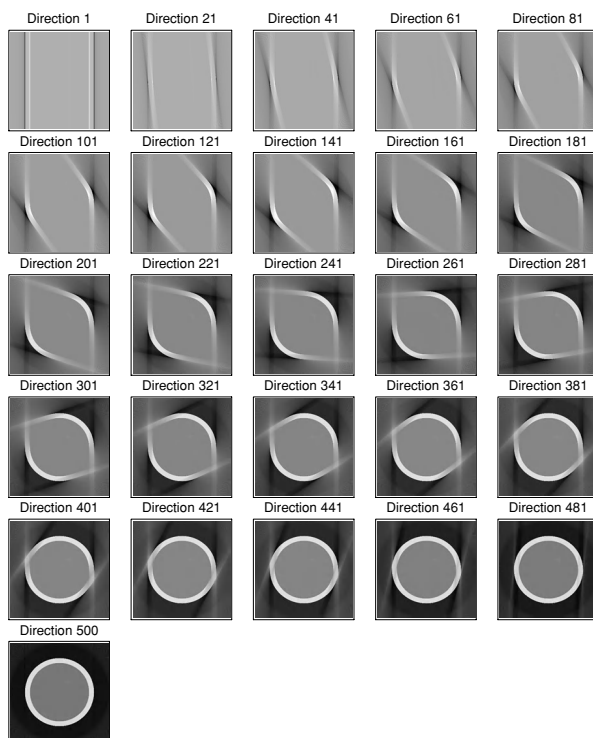


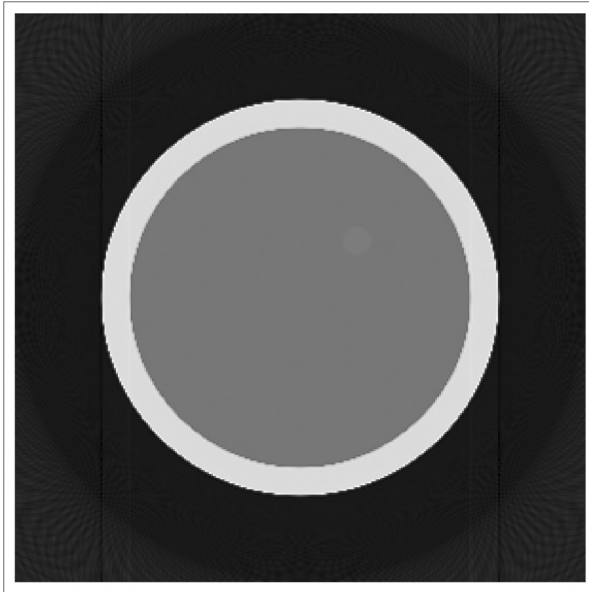
Figure 1.26. Snapshots of filtered backprojection applied to the x-ray data of the unknown object f .

as we enter the object the density is very high but quickly drops off to an intermediate value. Therefore we can guess that the object is some sort of thin ring of high density surrounding a disk of moderate density. However, it is impossible to predict the detailed structure of f merely by looking at the sinogram of the x-ray data. This will appear in a dramatic way later.

Now let us perform filtered backprojection to simulate how a CT scanner would determine the shape and density of the unknown object.

We do filtered backprojection with 500 equally spaced directions. As in ordinary backprojections, each new filtered backprojection is averaged into the preceding ones. The initial filtered backprojection is shown in figure 1.26. After each group of 20 successive backprojections we show a snapshot of the process (it would take too much room to show every step). For example, the fourth picture is the average of 61 backprojections for angles $0, \frac{\pi}{500}, 2\frac{\pi}{500}, \dots, 60\frac{\pi}{500}$. The fifth picture averages the next ten filtered backprojections into the projections already obtained. Thus, the fifth picture is the average of 81 filtered backprojections. The final picture is the average of 500 filtered backprojections and represents our reconstruction of the unknown object from its Radon transform. We denote the final picture by the symbol f even though it is only an approximation to the original object.

Here is a larger version of the reconstruction.



It seems that we were correct in our prediction of the shape of the object. We now show how this reconstruction compares with the original object and how it compares with the ordinary (unfiltered) backprojection. Figure 1.27 shows, left to right, a density plot of the original object, a density plot of the filtered backprojection reconstruction f , and a density plot of the reconstruction by ordinary backprojection.

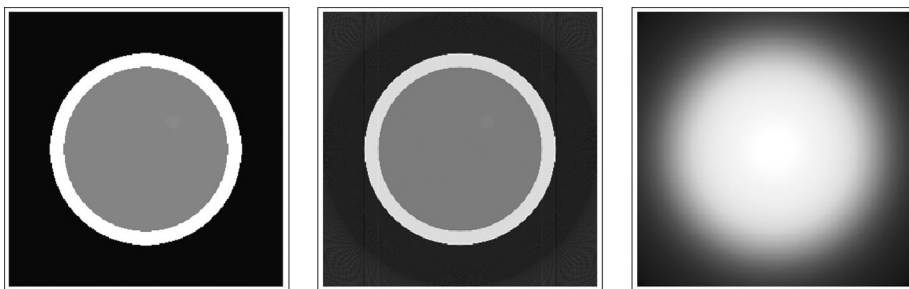
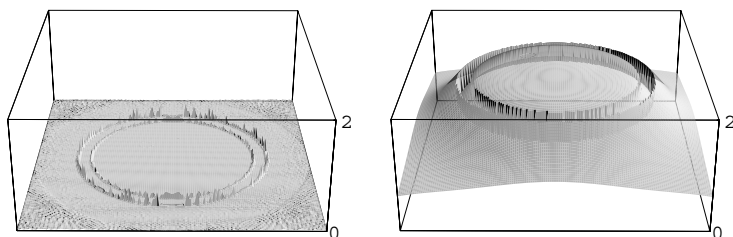


Figure 1.27. From left to right, original object, its reconstruction by filtered backprojection from 500 directions, and its reconstruction by ordinary backprojection from 500 directions.

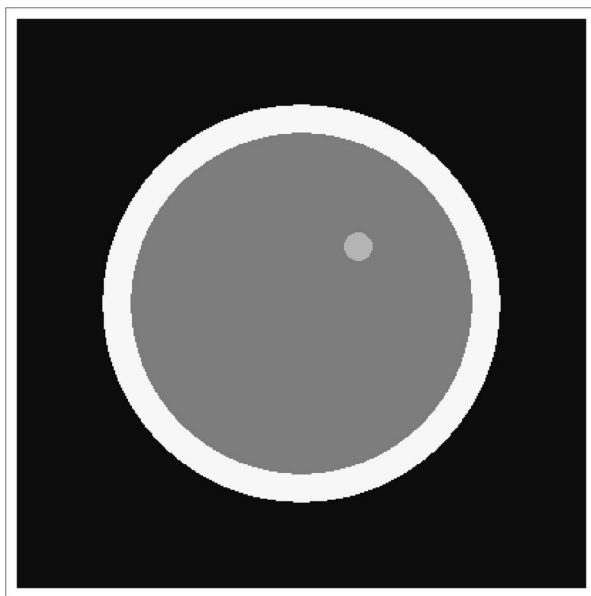
It appears that the filtered backprojection reconstruction f is a very good approximation to the original object. The ordinary backprojection shows the circular nature of the object, but it appears to be a very poor reconstruction. This is confirmed by viewing these objects as graphs. Actually what we do is present graphs of the differences between the objects: the smaller the difference, the better the approximation. In the next figure, the left-hand graph is the difference between the original object and its reconstruction by filtered backprojection. The right-hand graph is the difference between the original object and its reconstruction by ordinary backprojection.



We can see that filtered backprojection does a much better job.

The original object actually represents a mathematical model of a simplified human brain. Such a model is called a phantom. In this phantom the densities are those of the actual tissue in the brain. The densest tissue is that of the skull, represented by the white outer ring. The gray matter has somewhat less than half the density of the skull. In actual CT we are interested in locating pathologies. It is easy to pick up the difference between gray matter and the skull. But the difference in density between normal tissue and diseased tissue can be very small. For example, there is only a 4.5% difference in density between a metastatic breast carcinoma and gray matter in the brain, as compared with a 50% difference between gray matter and the skull. In fact, the subtle difference in density between the gray matter and a carcinoma is almost impossible to detect by

the unaided eye in the images we have been presenting. For that reason it would be interesting to perform an image enhancement process on the original object to enhance the contrast between normal and cancerous tissue. The next figure shows the original phantom after contrast enhancement.



It is now obvious that there is a small area of slightly higher density than gray matter in the upper right quadrant. Unfortunately, this is indicative of a carcinoma. It would be interesting to see if this is present in the reconstruction, because a radiologist would only have the aid of the reconstruction in making a diagnosis. Figure 1.28 shows, left to

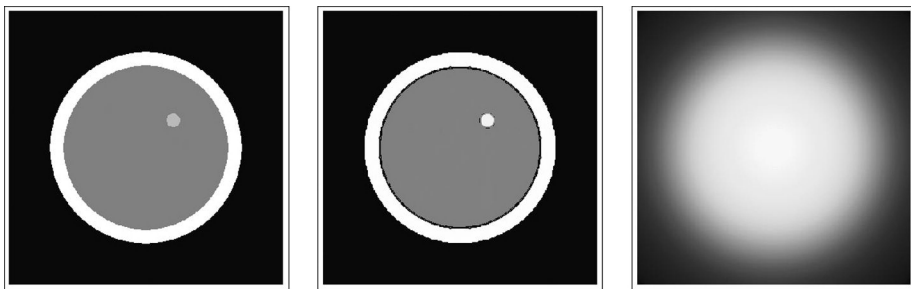


Figure 1.28. From left to right, original object, reconstruction by filtered backprojection, and reconstruction by ordinary backprojection with enhanced contrast.

right, density plots of the original object, the filtered backprojection reconstruction f , and the reconstruction by ordinary backprojection, all with contrast enhancement. We see that the reconstruction by filtered backprojection detects the carcinoma, whereas the ordinary backprojection does not. At this point we also can understand the comment made earlier that it is impossible to predict the detailed structure of f merely by looking at its x-ray data.

Go back to figure 1.27, which has a picture of the object before contrast enhancement. If you look carefully at the original object and its tomogram you may be able to see a faint trace of the carcinoma. This carcinoma is easily detected by the CT process, although the human eye requires some aid in the form of contrast enhancement to actually see it. This contrast enhancement is purely mathematical: it has no relation to the contrast-enhancing dyes used in some forms of radiology.

The carcinoma physically exists in the original object. Its effect on the Radon transform is very subtle and it would be very hard to detect in the sinogram, versus the same object without the carcinoma. The reason we cannot see the carcinoma with the naked eye is that there is such a small density difference between normal tissue and cancerous tissue. It is remarkable that this distinction can be detected by the process of tomography.

The contrast enhancement is not cheating, it merely emphasizes subtle differences in tissue. It is analogous to examining a picture in a dark room versus enhancing the picture with bright light from a lamp.

The next step is to see how well CT does with actual human organs.

1.7.2 Reconstruction of a Human Brain from X-ray Projections

We now present the reconstruction of an image of a human brain from its x-ray projections by the same filtered backprojection process used by CT scanners. The known data are the set of x-ray projections shown by the sinogram in figure 1.29, as may be collected by the detectors of a CT scanner.

Figure 1.30 shows the succession of filtered backprojections used in reconstructing the brain section. The x-ray projections were sampled in 1,000 equally spaced directions. The first backprojection and every fortieth backprojection thereafter are shown along with the reconstruction, which is the last backprojection (lower left).

Figure 1.31 presents a larger version of the reconstructed brain section.

We can show that this is a good reconstruction. First we compare the original brain image with the reconstruction (fig. 1.32).

In figure 1.33, we show graphs of the original brain section image, the filtered backprojection reconstruction, and the difference between the original image and the reconstruction.

We therefore see that the filtered backprojection process works very well, at least on objects no more complicated than human tissue.

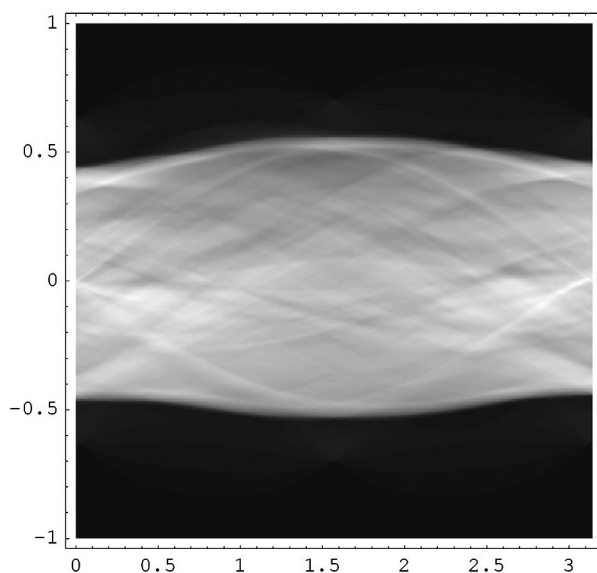


Figure 1.29. Sinogram showing the x-ray projections of a human brain.

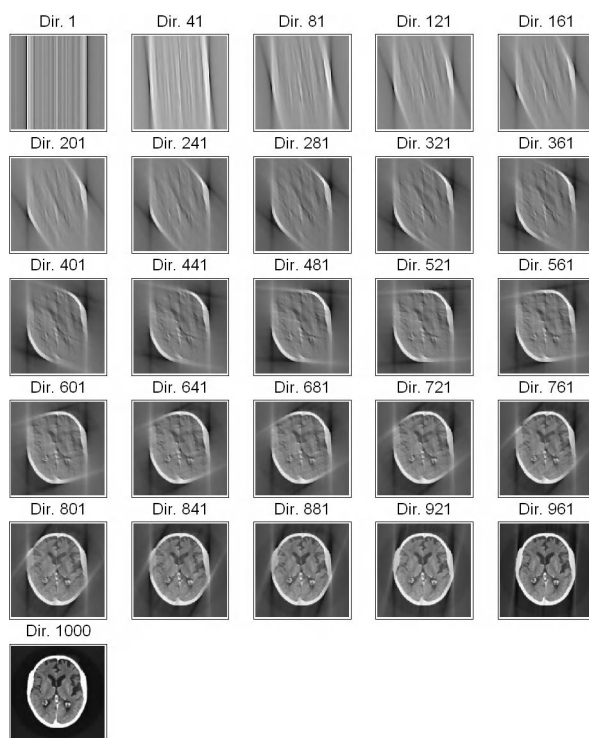


Figure 1.30. Snapshots of filtered backprojection reconstruction of a human brain (1,000 projections).

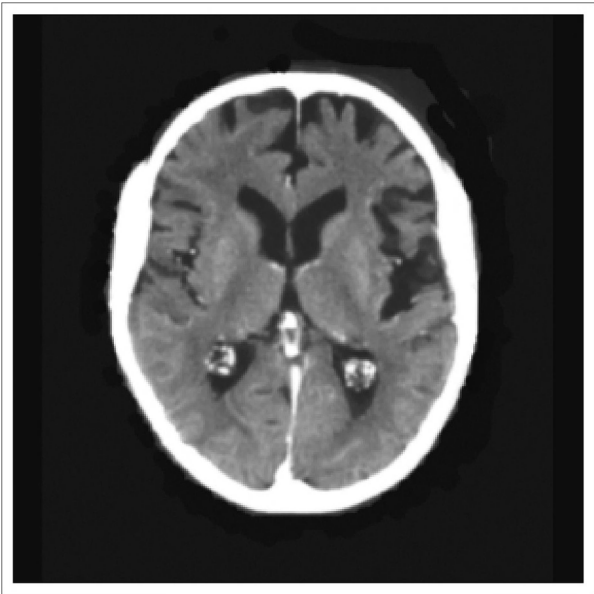


Figure 1.31. Filtered backprojection reconstruction of a human brain.

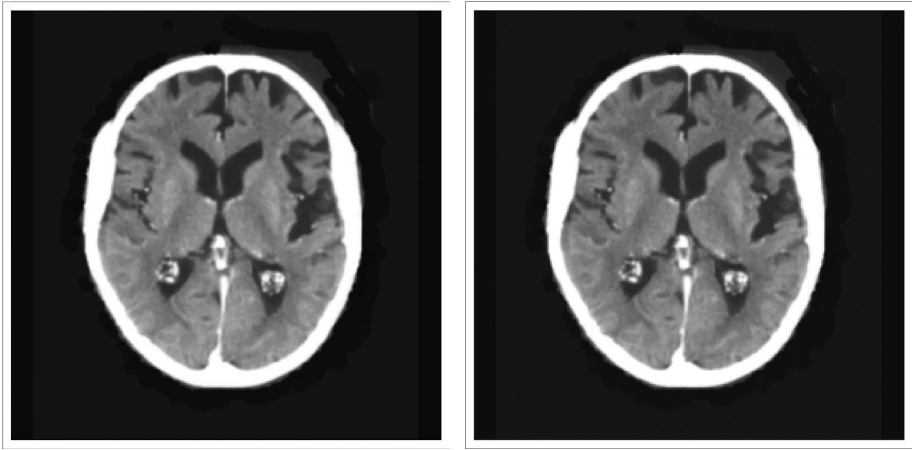


Figure 1.32. Comparison between original brain image and its filtered backprojection reconstruction. The original is on the left.

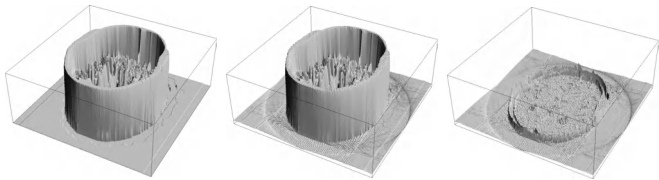
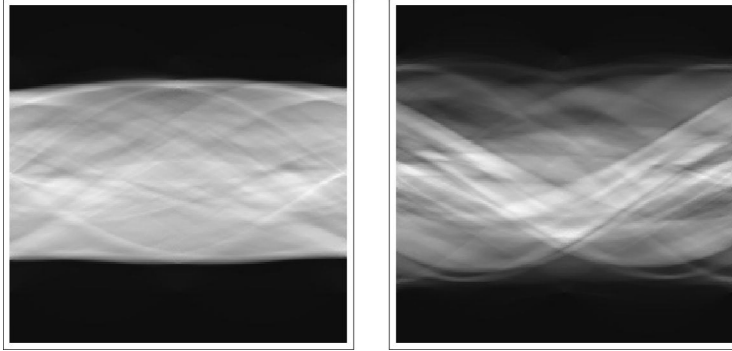


Figure 1.33. Another comparison. The original is on the left, the reconstruction is in the middle, and the difference between them is on the right.

1.8 Which Is Which?

We can now answer the question posed at the end of section 1.4 where we gave an example of x-ray data for a brain and an abdomen. Here are the sinograms representing these data.



At this point we recognize the left-hand sinogram as that of the brain section discussed in the previous section. Therefore, the right-hand sinogram must correspond to the abdominal section. Here is a picture of the reconstruction of that section. You can see quite clearly the spinal column and kidneys and other anatomical features.



Many interesting questions about the Radon transform remain to be answered, but we have gone as far as possible using only elementary mathematics.

1.9 How X-rays Determine Density

In this section we explain how an x-ray can determine the density of an object along a line. An x-ray beam can be considered to be a collection of energetic photons moving along a straight line. As the x-ray beam traverses tissue some photons may be scattered or absorbed. Heavier tissue tends to scatter or absorb x-rays more than lighter tissue. Therefore, it is possible to get a good estimate of the total density along the line traversed

by the x-ray beam by comparing the number of photons emitted by the x-ray source with the number of photons counted by the detector at the other end.

Actually the behavior of x-rays is more subtle. The attenuation of the x-ray beam due to absorption and scatter also depends on the energy of the beam and the atomic characteristics of the material being traversed. But the previous paragraph indicates that the total attenuation experienced by an x-ray beam traversing a line is informally related to the total density of the material along the line, that is, to the x-ray projection. At the risk of being slightly imprecise, we will treat the terms “attenuation” and “density” synonymously in this section.

This section is more technical than the rest of the chapter. However, I have tried to present the material so that a reader with just a small amount of mathematical knowledge can follow the argument. The only exception is a technical note that requires some knowledge of the calculus.

Figure 1.34 illustrates a classic experiment showing the phenomenon of *photon attenuation* in an x-ray beam. Identical plates consisting of a homogenous material are stacked together and an x-ray beam traverses the plates from left to right. As the beam exits each plate, the number of surviving photons decreases. Each plate can be examined to see the intensity of the x-ray beam as it exits the plate; in figure 1.34, note how fewer and fewer photons survive through each plate. Therefore, the number of photons that exit the last plate on the right is less than the number that entered on the left. This experiment was repeated by using many different materials for the plates. In each case it was observed that the number of photons that did not survive the trip over a small distance d is a fraction of the number beginning the trip. However, over a large distance, the relationship between the number of photons entering the material and the number exiting is more intricate. To investigate this relationship we introduce the idea

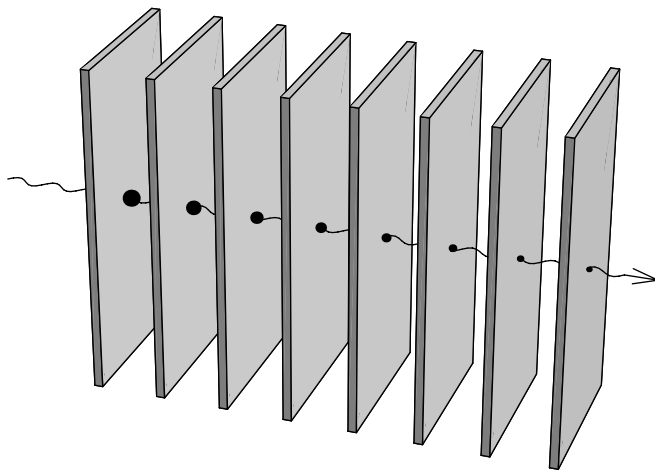


Figure 1.34. Photon attenuation in an x-ray beam.

of the **attenuation coefficient** of the material in question.¹⁴ The attenuation coefficient is denoted by the symbol μ (μ is the symbol for the Greek letter mu). It is defined in the following way. Take the fraction of photons that did not survive the trip through a small distance d of the material. Then divide this fraction by d . The resulting quantity, μ , is the attenuation coefficient of the material. For example, if we observe that 10,000 photons enter a 0.1-*cm* length of some material and that 9,500 photons exit, then the fraction of photons that did not survive is

$$\frac{10,000 - 9,500}{10,000} = 0.05$$

If we now divide this fraction by the small distance $d = 0.1$ *cm* we arrive at the quantity μ :

$$\begin{aligned}\mu &= \frac{0.05}{0.1} \\ &= 0.5 \text{ photons/cm}\end{aligned}$$

If we perform the same experiment over half the distance we would observe that 9,750 photons survive the distance of 0.05 *cm*. Therefore the attenuation coefficient computes to

$$\mu = \frac{10,000 - 9,750}{10,000} \div 0.05 = 0.5$$

and we see that we get the same attenuation coefficient. The experiment illustrated in figure 1.34 leads to the Lambert-Beer law. One aspect of the Lambert-Beer law is that the attenuation coefficient is constant for homogeneous materials. Of course each material has a different attenuation coefficient.

To obtain the Lambert-Beer law, we need to use the idea of the attenuation coefficient in the following way. From the definition,

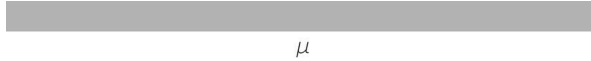
$$\mu d = \text{fraction of surviving photons},$$

provided the distance d is small. For example, if we have a material whose attenuation coefficient is $\mu = 0.3$ and if an x-ray beam traverses a length $d = 0.01$ of this material, then $\mu d = 0.003 = \frac{3}{1,000}$, so 3 of every 1,000 photons will be blocked while traversing the distance d . This can be phrased in terms of percentage: the fraction of surviving photons is $1 - \mu d = \frac{997}{1000}$ and this means that 99.7% of the photons will survive the passage of the small distance d .

Let N_0 denote the number of photons entering the material and let N_1 denote the number of photons exiting. In the preceding example where $\mu = 0.3$ and the photons traversed a distance $d = 0.01$ before exiting, we had $N_0 = 1,000$ and $N_1 = 997$.

¹⁴ This result holds only for monoenergetic x-ray beams. The x-ray beams used in CT scanners actually consist of photons of many different energies. The attenuation coefficient actually depends on the material and the energy of the photons. However, there are ways of resolving this problem, and it is safe for the current discussion to assume that the attenuation coefficients depend only on the atomic number of the material.

We are more interested in what happens to N_1 when an x-ray beam traverses a long distance through inhomogeneous material. To investigate this, let us take a simple case in which an x-ray beam traverses a line containing 10 cm of tissue with density μ . The situation is depicted in the following figure.



In this investigation we use the law of exponents which states that

$$x^a x^b = x^{a+b}$$

and the fact that if

$$x^a = c$$

then

$$a = \log_x c$$

(i.e., the logarithm to the base x of a positive number c is precisely the power a such that $x^a = c$).

To continue this investigation, we need a simple mathematical relationship. Define the real number $g = 0.3679$ (those readers who have studied the calculus can check that g is approximately e^{-1}). The significance of this number is that it allows one to express the difference $1 - x$ as a power of g . That is, $1 - x \approx g^x$, if x is relatively small. This ability to express $1 - x$ as a certain power of g is advantageous later. When we use this relationship we can substitute any small number for x . For example if μ_1 is a fraction and d is a short distance, then the relationship $1 - x \approx g^x$ implies that $1 - \mu_1 d \approx g^{\mu_1 d}$. We will use this type of calculation several times.

This approximation can be verified in a completely rigorous way using the calculus, but let us illustrate the idea using a few values of x

x	g^x	$1 - x$
0.01	0.990 05	0.99
0.05	0.951 23	0.95
0.09	0.913 94	0.91

From this table we see evidence that the relationship $g^x \approx 1 - x$ is valid, at least if x is relatively small.

Remember that we are trying to see how many photons survive the trip through the 10-cm piece of material. N_0 photons enter at the left and N_1 , the number of surviving photons, exit at the right.

To see what happens to photons traversing long distances, we need to break up a long distance into small pieces. We do this by breaking the entire 10-cm line into small pieces of size d . The actual length of d is not important, any small length will do. From the Lambert–Beer law we know that the photons that successfully pass through the first small segment of length d will contain only the fraction $(1 - \mu d)$ of the photons that

were initially in the beam. The real number $(1 - \mu d)$ is always between 0 and 1 and hence is a fraction. It helps in the following discussion to think of the abstract fraction $(1 - \mu d)$ as a definite fraction such as $\frac{3}{100}$ or 0.456. Hence, multiplying a quantity N_0 by $(1 - \mu d)$ will always result in a smaller quantity N_1 .

In particular, if we start with N_0 photons entering the segment, only $N_0(1 - \mu d)$ will exit. We use the approximation, above with $x = \mu d$, to get

$$N_0(1 - \mu d) \approx N_0 g^{\mu d}$$

as the number of photons, out of the initial number N_0 , that have survived the passage over the segment of length d . Over the second segment of length d only the fraction $(1 - \mu d)$ of the $N_0 g^{\mu d}$ photons that enter will survive. The approximation then yields

$$(1 - \mu d)(N_0 g^{\mu d}) \approx N_0 g^{\mu d} g^{\mu d} = N_0 g^{2\mu d}.$$

In the same way, over the third small segment of length d we can check that approximately $N_0 g^{3\mu d}$ photons will survive. In general, we will have $N_0 g^{x\mu d}$ photons surviving past x adjacent small segments of length d . This pattern will persist until the beam traverses the entire 10-cm length of material. We note that after the last segment we have $x = \frac{10}{d}$, because, in a 10-cm line segment, there are exactly $\frac{10}{d}$ pieces of length d . At this point the surviving number of photons, which by definition is N_1 will be about

$$N_1 = N_0 g^{\left(\frac{10}{d}\right)\mu d} = N_0 g^{10\mu}$$

(Note how, in the end, there is no dependence of the answer on the particular length d .) Hence, we get $g^{10\mu} = \frac{N_1}{N_0}$. Take the logarithm to base g of this expression to get

$$10\mu = \log_g \left(\frac{N_1}{N_0} \right).$$

But the 10-cm line, being homogeneous, has a total density of $10\mu/\text{m}$ so for homogeneous lines we get

$$\text{Total density} = \log_g \left(\frac{N_1}{N_0} \right).$$

If we have a line containing a nonhomogeneous amount of material, then we break the line up into pieces that are approximately homogeneous. We illustrate this with the simple case of a line containing 10 cm of tissue with density μ_1 followed by 5 cm of tissue with density μ_2 as depicted in the following figure.



If N_0 photons enter at the left, then, as we saw above, $N_0 g^{10\mu_1}$ will survive the passage through the first 10-cm piece with density μ_1 . We then use this number, $N_0 g^{10\mu_1}$, as the input number of photons to the 5-cm piece of density μ_2 . This results in $N_1 = (N_0 g^{10\mu_1}) g^{5\mu_2}$ as the number of photons which survive the entire passage. We can use the law of exponents to simplify this expression, thus arriving at

$N_1 = N_0 g^{10\mu_1 + 5\mu_2}$ and, hence, $g^{10\mu_1 + 5\mu_2} = \frac{N_1}{N_0}$. As before we take logarithms to get $10\mu_1 + 5\mu_2 = \log_g(\frac{N_1}{N_0})$. But the total density along the line is $10\mu_1 + 5\mu_2$, so again we arrive at

$$\text{Total density} = \log_g \left(\frac{N_1}{N_0} \right). \quad (1.1)$$

This result can be extrapolated to any number of adjacent homogeneous pieces. This is accurate to a good approximation for any piece of tissue that we may encounter.

Because CT scanners can produce these photon counts, we have solved the problem of how an x-ray beam can give the total density of material lying along a line. We use the CT scanner to obtain the number N_o (of photons emitted from the x-ray source) and the number N_1 (of photons that have survived to be counted at the detector) along a given line. We substitute these numbers into equation (1.1) which then provides us with the total density, or x-ray projection, along that line.

Those readers who have not studied the calculus can skip the following technical note.

Technical Note 1. *For those readers who know some calculus, we can derive the exact relationship between the line integral of the attenuation coefficients and the photon counts along the path of a monoenergetic x-ray beam. The **intensity of the beam** is the photon count at any point of the beam and we denote it by N . The attenuation of a monoenergetic x-ray beam is governed by the Lambert-Beer law, which is illustrated by the experiment shown in figure 1.34. The result of this experiment is that the attenuation of the x-ray beam is exponential relative to the distance traversed. More precisely, if an x-ray beam traverses a homogeneous material M along a straight line L , parametrized by distance s , then the intensity N of the beam at a distance s from the origin is given by*

$$N(s) = e^{-\mu s} \quad (1.2)$$

for some positive constant μ . This relationship has been experimentally verified for many different materials and we accept it as an axiom. Equation (1.2) is the global version of the Lambert-Beer law. We can easily obtain a local version by differentiation:

$$\frac{dN}{ds} = -\mu N \quad (1.3)$$

Going back to the experiment with the photographic plates from which we deduced the Lambert-Beer law, it is a reasonable assumption that (1) The experiment is independent of the direction of the line L : if we rotate the apparatus, then the same exponential law will arise, independent of the direction of rotation. (2) The local version of the Lambert-Beer law should hold for a nonhomogeneous medium as long as the intensity function is reasonably smooth.

Therefore, the constant μ in the Lambert-Beer law, at each point, is a property of the material and is independent of the position or orientation of the material, so it is reasonable to define a function $\mu = \mu(x, y)$ whose value at the point located at (x, y) is the attenuation coefficient of the material located at that point. Furthermore,

the local form of the Lambert-Beer law is obeyed by μ . This means that if ds is the element of length along any line in the plane and if an x-ray beam is sent along this line, then the intensity of the beam satisfies the same differential equation as before:

$$\frac{dN}{ds} = -\mu N \quad (1.4)$$

The function μ is called the **linear attenuation coefficient** of the tissue (or other material) being scanned.

Now let L be a line passing through the unknown object. We assume that the x-rays are being generated at a fixed point on the line exterior to the unknown object. We let N_0 be the number of photons that are produced by this x-ray source. Likewise an x-ray detector has been placed at another fixed point on the line and outside the unknown object. We let N_1 denote the number of photons that have survived the passage along L and are counted by this detector. By integrating equation (1.4) and using a simple change of variable from s to N we see that

$$\begin{aligned} \int_L \mu(s) ds &= \int_{N_0}^{N_1} -\frac{1}{N} dN \\ &= \ln\left(\frac{N_0}{N_1}\right) \end{aligned} \quad (1.5)$$

Hence, the line integral of μ along L (i.e., the total density of μ along L) may be obtained directly from the photon counts according to equation (1.5). Therefore the collection of these integrals for all lines L in the plane form the Radon transform of μ . This is pursued in more detail in chapter 2.

Therefore, a CT scanner actually tries to reconstruct the linear attenuation coefficient function, not the density function. However, it is reasonable to suppose that a close connection exists between the attenuation coefficient and the density of the object. After all, the more matter present, the more chance there is to absorb or scatter a photon.

In practice there are some difficulties with this basic idea. The main difficulty is that the x-ray beams used in CT consist of photons at many different energy levels; they are not monochromatic as assumed above. Therefore, the attenuation coefficient is a function $\mu = \mu(x, E_0)$ of both position and the energy of the photon. A related problem is beam hardening, which means that less energetic photons tend to be absorbed or scattered earlier, thus causing the composition of the x-ray beam to consist of more energetic photons further down the line. Fortunately, there are corrective algorithms that take the number of initial photons and the number of exiting photons as input and can give as output a good estimate for $\ln\left(\frac{N_0}{N_1}\right)$ at a certain fixed energy level E_0 . From this we can compute a good approximation to $\int_L \mu(x, E_0) dx$. Finally, radiologists are skilled in correlating the attenuation coefficients with the actual structure of the tissue, so CT scans turn out to be reliable diagnostic tools.

Herman's book [296] and the article by Cormack and Quinto [109] have more details on the interaction of radiation with matter.

1.10 Additional References and Results

1.10.1 History of Tomography in Medical Diagnosis

The most extensive history of tomography in medical diagnosis is contained in Webb's book [618]. The outline of this history that I have given here is derived mainly from [618] with supplementary information from Natterer [444], Natterer and Wübbeling [446], Herman [296], Cormack [104], and personal communications with colleagues.

The use of tomography in medicine is broadly classified into "classical tomography" and "computerized tomography." In classical tomography the desire is to image a two-dimensional section of the body. X-rays are transmitted through a neighborhood of this section in such a way as to produce as sharp a focus as possible on the desired section while blurring the surrounding tissue. This method was fairly popular in the early part of the twentieth century, but the results were of debatable resolution. One expert claimed that this method was "...an expensive way to obtain bad radiographs." (Webb [618], page 107). Because this method only shares its name with computerized tomography and does not utilize the Radon transform, we will not spend any more time in describing it. Readers interested in further details are referred to Webb [618].

According to Webb [618] there are "at least two (Frank and Takahashi) and possibly more pioneers whose work in the 1940s might be viewed as a direct precursor of CT." Frank was awarded a German patent for his ideas, but he used pure backprojection, which does not give reasonable reconstructions, as we saw in section 1.5. Kuhl and Edwards [366] also used backprojection with no better results.

Also according to Webb [618], the earliest implementation of computerized CT was developed by Korenblyum, Tetelbaum, and Tyutin [357] at the Kiev Polytechnic Institute in 1958. Their reconstruction was based on Tetelbaum's paper [599], which gave a method for inverting the Radon transform. In their paper [357] Korenblyum, Tetelbaum, and Tyutin outlined the mathematical procedure for fan beam scanning and described an experiment in which a 100×100 pixel image was reconstructed from x-ray projections. Their method involved recording the sinogram on film and then producing electrical signals from this film which were processed by an analogue computer. Webb [618] has diagrams of their apparatus. Korenblyum, Tetelbaum, and Tyutin described their work as experimental, and apparently, it was never clinically implemented. As of 1983, no further evidence of publications on CT were found in the Russian literature.

The Nobel prize for Physiology or Medicine was awarded jointly to A. M. Cormack and G. N. Hounsfield in 1979. Each of these laureates made separate and important contributions to the field of computerized tomography. The presentation speech for the 1979 Nobel prize for Physiology or Medicine was delivered by Professor Torgny Greitz of the Karolinska Medico-Chirurgical Institute. He stated that

Cormack realized that the problem of obtaining precise values for the tissue-density distribution within the body was a mathematical one. He found a solution and was able, in model experiments, to reconstruct an accurate cross-section of an irregularly shaped object. This

was reported in two articles, in 1963 and 1964. Cormack's cross-section reconstructions were the first computerized tomograms ever made – although his “computer” was a simple desktop calculator.¹⁵

Greitz's comment about Cormack having made the first computerized tomograms may not be historically accurate (compare, the paragraph on Korenblyum, Tetelbaum, and Tyutin, above). However, the early Russian tomography never advanced beyond the simple experiment of Korenblyum and colleagues. On the other hand, Greitz went on to say:

Publication of the first clinical results in the spring of 1972 flabbergasted the world. Up to that time, ordinary X-ray examinations of the head had shown the skull bones, but the brain had remained a gray, undifferentiated fog. Now, suddenly, the fog had cleared. Now, one could see clear images of cross-sections of the brain, with the brain's gray and white matter and its liquid-filled cavities. Pathological processes that previously could only be indicated by means of unpleasant – indeed, downright painful and not altogether risk-free examinations could now be rendered visible, simply and painlessly – and as clearly defined as in a section from an anatomical specimen. . . . Allan Cormack and Godfrey Hounsfield! Few laureates in physiology or medicine have, at the time of receiving their prizes, to the degree that you have, satisfied the provision in Alfred Nobel's will that stipulates that the prizewinner “shall have conferred the greatest benefit on mankind.” Your ingenious new thinking has not only had a tremendous impact on everyday medicine; it has also provided entirely new avenues for medical research. It is my task and my pleasure to convey to you the heartiest congratulations of the Karolinska Institute and to ask you now to receive your insignia from His Majesty, the King.

Hounsfield, who worked for the EMI company in England, designed the first commercially available CT scanner in 1972. However, he traveled a long road to produce this machine. Hounsfield started his research in 1967. His initial experiment required nine days of radiation exposure to gather enough data to reconstruct one picture (I believe he used a cow's brain for this experiment). Then it required two and a half hours of computing time to produce the reconstructed image. By 1972 when EMI began marketing Hounsfield's CT scanner, the data collection time had been considerably reduced and it took only four and a half minutes of computing time to produce an image. Modern CT scanners work much faster still.

1.10.2 Further References

I can not make many suggestions for a reader who has no training in mathematics. One possibility is Cormack's article [102], which is very interesting and which describes his early research leading to his Nobel prize.

A few references are accessible to readers with a good background in undergraduate mathematics. Panton's 1981 article [468] still makes for very interesting reading. He discusses the history of tomography and graphically illustrates the backprojection

¹⁵ The quotations by Greitz are from Nobel Lectures, Physiology or Medicine 1971–1980. Edited by Jan Lindsten. Singapore: World Scientific Publishing Co., 1992.

process. The remainder of his paper requires some advanced undergraduate mathematics. Other articles of interest are Shepp and Kruskal [556], Herman [296], Zalcman [625], Strichartz [585], Nievergelt [449,450], Deans [124], Kak and Slaney [328], Natterer and Wübbeling [446], and Epstein [150]. Also the next chapter of this book, which presents the mathematical derivation of the properties of the Radon transform, should be accessible to readers with a good background in undergraduate mathematics.

We should also mention the two interesting papers [400,401] by Louis, both with the title “Medical imaging: state of the art and future development.” The first paper [400] is fairly elementary from the mathematical point of view but has very interesting informal descriptions of the process of data collection and reconstruction of images for medical diagnosis. This paper should be accessible to the reader with a good background in university-level mathematics. The second paper requires a more substantial mathematical background. It gives a very detailed overview of both the process and the mathematics of various forms of medical tomography. The emphasis in [401] is practical, with attention being paid to the development of algorithms for medical tomography. For the reader interested in the practical side of tomography, the articles [400,401] also provide an extensive list of references dealing with applications to medical imaging.

Hochbruck and Sautter have a fairly elementary article [313], in German, which has some interesting pictures of CT reconstructions. They also give a derivation of the Lambert-Beer law using a half-life argument. We assumed the Lambert-Beer law based on experimental evidence. The remainder of their article gives an algebraic reconstruction (ART) method for CT. ART methods are fairly simple but are not used much in CT. However, ART techniques are used in a different method of tomography called SPECT. In Chapter 5 we give a derivation of a generalized Radon transform which models SPECT tomography. However, we do not pursue ART methods, because they have little connection to mathematical analysis.

The next set of references require the reader to have had at least some graduate level mathematics courses.

First, I should mention the book by Gel’fand, Gindikin, and Graev [189], which is an almost breathtakingly beautiful introduction to the subject of mathematical tomography, written by some of the pioneers in the field.

The following books and articles delve deeply into various areas of the subject and require a substantial background in mathematics: Ehrenpreis [149], Gel’fand and Shilov [195], Gel’fand, Graev, and Vilenkin [194], Helgason [291], Natterer [444], Natterer and Wübbeling [446], Smith, Solmon, and Wagner [566], Kak and Slaney [328], and Ramm and Katsevich [513]. The articles [148, 147] by Ehrenpreis are a suggested accompaniment to reading his book [149].

The book by Lavrent’ev and Savel’ev [377] is a good reference for some of the functional analytic tools used in tomography.

The book by G. T. Herman [296] contains a wealth of information on the relation between line integrals and x-ray counts, photon statistics and data collection for CT, including correction schemes for beam hardening and polychromaticity. The book by S. R. Deans [124] also has a discussion of the relation between line integrals and x-ray counts. Both these references have many examples of practical uses of tomography

besides CT. For a how-to guide to the practical side of tomography, consult Herman [296], Kak and Slaney [328], or Epstein [150]. The books by F. Natterer [444] and Ramm and Katsevich [513] have briefer derivations of the relation between line integrals and x-ray counts and briefer introductions to the applications but are valuable for the theoretical treatment of tomography.

There are a few survey articles accessible to a reader with a good background in graduate-level mathematics. Of particular note are the appendix [364], written by Kuchment and Quinto, of the Ehrenpreis book [149], the article [158] by Faridani titled “Introduction to the Mathematics of Computed Tomography,” which appears in the book [606] edited by Uhlmann and the article by Smith, Solmon, and Wagner [566].

There are many references that give a very brief introduction to the Radon transform and its inversion. The translation of the Soviet ‘Mathematical Encyclopedia’ [612] (Vol. 9, Sto-Zyg, pp. 179–181) has an excellent short introduction to the Radon transform and tomography. Also, Khavin and Nikol’skij [351] have a one-paragraph treatment of the Radon transform; J. S. Walker [616] gives a succinct, somewhat specialized introduction to Radon transform in an appendix (pp. 380–400); Terras [597] defines the Radon transform early in the book (pp. 107–119) and has very succinct guided exercises in which the reader can develop Radon inversion; volume two of Terras [598] has a brief mention of applications to partial differential equation; and Dym and McKean [138] provide a short introduction to the Fourier inversion method for the Radon transform.

A few applications of Radon transforms are discussed in chapter 2, section 2.9.

1.10.3 Notes

Tomography and the theory of the Radon transform lie at the intersection of the mathematical fields of integral geometry and inverse problems. An area of integral geometry is concerned with geometric properties of objects that are determined by some set of integrals over portions of the objects. If these portions are straight lines, then we arrive at the Radon transform and hence tomography. The book by Gardner [185] contains a wealth of information on the integral geometric aspects of tomography. He also has a brief introduction to some aspects of analytic tomography. Also of interest in reference to integral geometry are the books of Nachbin [433], Santaló [543], and Ren [515].

To understand what is involved in the mathematical field of inverse problems, let us first describe the idea of a direct problem. A direct problem takes some given (input) data, performs some well defined operations on these data, and produces the results of this computation as output. We can symbolize a direct problem in the following way: let f represent the input data, let A represent the calculations to be performed on the input, and let g represent the output. The direct problem can be written

$$g = A(f)$$

A simple example of a direct problem is the computation of the values of a polynomial: if P is the polynomial $P(x) = x^2 + 1$, then we define $A(f) = P(f)$. For example, $A(2) = 2^2 + 1 = 5$. A more interesting direct problem is that of computing the Radon

transform of a given function:

$$g(\theta, p) = Rf(\theta, p)$$

This means that we are given complete information about the object f and we must compute all the x-ray projections of f .

An inverse problem can be written in the same form

$$g = A(f)$$

However, the roles of f and g are interchanged. Now it is g which is the input data and we are required to find f . In the case of the polynomial problem, a typical inverse problem would be to solve for x given that

$$17 = A(x) = x^2 + 1.$$

Solving this inverse problem would require finding all the solutions x to this equation. Likewise, the inverse Radon problem is to determine an unknown object by using its Radon transform as the input data.

The Radon Transform

2.1 Introduction

In chapter 1, we saw that the Radon transform is a fundamental tool for using x-rays in medical imaging. The value of the Radon transform of a function f on a line L was defined to be the total density of f along the line and could be represented as an integral of the form

$$\int_L f(x) dx \tag{2.1}$$

(Recall the discussion of x-ray projections and the Lambert-Beer law, chapter 1, section 1.9.) Furthermore, we described tomography as the study of reconstructing objects from their x-ray projections. Therefore, we can rephrase this by saying that *tomography* is the process of reconstructing an unknown function from its Radon transform.

In this chapter we rigorously define the Radon transform and develop its elementary properties. Consequently, we will be able to give a mathematical explanation for the filtered backprojection reconstruction method that was introduced in Chapter 1. We end the chapter with Radon's original inversion formula from his 1917 paper [508].

Because the field of tomography is very broad, and to accommodate those readers who are not mathematicians, I have tried to make this chapter accessible to readers who have had at least university-level courses in the calculus and linear algebra. These readers will be pleasantly surprised to learn that most of the basic theory of tomography follows from only two theorems that they will have studied in the calculus: the theorem on change of variables in a multiple integral and the theorem on iterating a multiple integral. However, all results are precisely stated, in n dimensions, and the proofs are mathematically rigorous.

There are practical and theoretical applications of the Radon transform on spaces of higher dimension than the plane, and with just a little effort we can extend the definition and study of the Radon transform to higher dimensions. Looking at the integral defining the Radon transform, equation (2.1), we see that an extension to higher dimensions can be accomplished by replacing the line L with some other domain. Therefore, we need

to study integration on these domains. Let us begin by examining the mathematical description of lines in a plane.

A line in the plane is the locus of points described by a linear equation of the form $a_1x_1 + a_2x_2 = c$. The **Euclidean plane** $\mathbb{R}^2$ consists of all points x of the form (x_1, x_2) , where x_1 and x_2 are real numbers. We use the terms “point” and “vector” interchangeably because they are the same from the mathematical point of view. The inner product in $\mathbb{R}^2$ is denoted by $\langle a, x \rangle = a_1x_1 + a_2x_2$ for any two points $a, x \in \mathbb{R}^2$ (the expression $x \in A$ means x is an element of the set A or, equivalently, that x is in the set A). The inner product is sometimes called the dot product and is also denoted by $a \cdot x$. We can therefore rewrite the equation of the line in the form $\langle a, x \rangle = c$. Analytic geometry shows that this gives a line orthogonal to the vector $a = (a_1, a_2)$. When this equation is multiplied by a nonzero constant, we still get the same line. To normalize the situation we can demand that the vector a be a unit vector. It is traditional to denote unit vectors by the symbol θ and we denote the set of all unit vectors by S^1 , which is called the **unit sphere** of $\mathbb{R}^2$. The unit sphere of $\mathbb{R}^2$ is just a fancy name for the circle of radius 1 centered at the origin. Shortly we will introduce unit spheres in any Euclidean space $\mathbb{R}^n$.

In the case that a is a unit vector θ , the equation of the line looks like

$$\langle \theta, x \rangle = c \quad (2.2)$$

and this determines the line orthogonal to the unit vector θ , which lies at a distance c from the origin. Also, if we are given any line in the plane that lies at a distance c from the origin, then there are precisely two unit vectors θ which satisfy the defining equation (2.2). If θ is one of these vectors then the other is $-\theta$ and the two corresponding equations are of the form:

$$\begin{aligned} \langle \theta, x \rangle &= c \\ \langle -\theta, x \rangle &= -c \end{aligned}$$

From this we see that the distance c is a signed distance and can be any real number. In other words, specifying a unit vector θ and a real number c uniquely determines a line in the plane (although the vector and distance are not uniquely determined by the line).

From this discussion it is easy to generalize from the plane to n -space. Before doing this, note that we use the standard notation $\mathbb{R}$ for the set of real numbers and $\mathbb{C}$ for the set of complex numbers. Here is how we define Euclidean n -space:

Definition 2.1. $\mathbb{R}^n$ is the set of all ordered n -tuples of real numbers and is called **Euclidean n -space**. More precisely

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : \text{each } x_j \text{ is a real number}\}$$

Remark 2.2. Here we use the set descriptor notation: a set S which is characterized by a property P is written as $S = \{x : x \text{ has the property } P\}$. This is read as “the set of all x such that x has the property P ”.

The next definition is related and useful.

Definition 2.3. Given two sets A and B , the set of all ordered couples (a, b) , where $a \in A$ and $b \in B$ is called the **Cartesian product** of A and B and is denoted by $A \times B$. Using ordered n -tuples we generalize this to

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) : x_j \in A_j, j = 1, \dots, n\}$$

In the special case that each $A_j = \mathbb{R}$ we get Euclidean n -space and it makes much more sense to write this as $\mathbb{R}^n$ rather than $\overbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}^{n\text{-factors}}$

Example 2.4. $\mathbb{R}^1$ consists of all one-tuples, points of the form (x) , where x is a real number. Therefore, except for the difference in notation, “ x ,” versus “ (x) ,” the real number system $\mathbb{R}$ and Euclidean 1-space $\mathbb{R}^1$ are identical.

Euclidean 2-space $\mathbb{R}^2$ is the set of ordered 2-tuples (more commonly known as ordered pairs or couples) of the form (x, y) where x, y are real numbers and Euclidean 3-space $\mathbb{R}^3$ is the set of ordered 3-tuples (triples) of the form (x, y, z) where x, y, z are real numbers. Therefore $\mathbb{R}^2$ represents the Euclidean plane and $\mathbb{R}^3$ represents the usual 3-dimensional Euclidean space. In higher dimensions, a typical point of $\mathbb{R}^n$ is denoted by the ordered n -tuple $(x_1, x_2, \dots, x_n)$ where $x_1, x_2, \dots, x_n$ are real numbers.

Remark 2.5. The symbol $\in$ denotes set membership, so writing $w \in S$ is the same as saying that w is an element of the set S . Therefore the expression $x \in \mathbb{R}^n$ means that x is a point of the Euclidean space $\mathbb{R}^n$ and hence there exist real numbers $x_1, x_2, \dots, x_n$ such that $x = (x_1, x_2, \dots, x_n)$. The context will always determine whether a particular variable is a vector in $\mathbb{R}^n$ or a real number. In this way we avoid the need to write vectors in boldface type or with arrows over them. Furthermore, this convention has the advantage of allowing one to see the similarities in the vector space structure of the real numbers and the Euclidean spaces.

In short, $x \in \mathbb{R}^n$ signals that x is a point or vector in n -dimensional Euclidean space.

Each vector in $\mathbb{R}^n$ can be assigned a length. This is done via the idea of inner product:

Definition 2.6. The **inner product** of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$ or $\langle x, y \rangle$ and is defined by

$$\langle x, y \rangle = x \cdot y = \sum_{j=1}^n x_j y_j$$

We denote the **norm** or **length** of a point $x \in \mathbb{R}^n$ by $|x|$ and define it by

$$\begin{aligned} |x| &= \sqrt{\langle x, x \rangle} \\ &= \sqrt{\sum_{j=1}^n x_j^2} \end{aligned}$$

The norm is a generalization of the idea of absolute value and in fact the norm of x and the absolute value of x coincide when $n = 1$.

In the case of $\mathbb{R}^n$, the set of unit vectors is denoted by S^{n-1} , which is called the **unit sphere** of $\mathbb{R}^n$. Concerning the sphere S^{n-1} , in $\mathbb{R}^2$ this is a circle, which has dimension 1, whereas $\mathbb{R}^2$ has dimension 2. In $\mathbb{R}^3$ the unit sphere is a usual sphere and has dimension 2, whereas $\mathbb{R}^3$ has dimension 3. In general the unit sphere of $\mathbb{R}^n$ has one less dimension than $\mathbb{R}^n$ itself: the dimension of S^{n-1} is $n - 1$. This explains the superscript in the notation S^{n-1} . Regarding notation, θ will generally denote a point on the unit sphere of $\mathbb{R}^n$, not a real number (angle). However, on occasion we may use θ as a real number. As always, the context will clarify the usage.

2.1.1 Hyperplanes

A hyperplane is defined to be a set of points in $\mathbb{R}^n$ which is orthogonal to a specific unit vector θ and which lie a directed distance c from the origin. The precise definition is:

Definition 2.7. A **hyperplane** in $\mathbb{R}^n$ is the set of points $x \in \mathbb{R}^n$ satisfying an equation of the form

$$\langle \theta, x \rangle = c$$

where $\theta \in S^{n-1}$ is a unit vector and where c is a real number.

This definition makes sense. For example, in $\mathbb{R}^3$ if $\theta = (a_1, a_2, a_3)$, then the defining equation is of the form $\langle \theta, x \rangle = a_1x_1 + a_2x_2 + a_3x_3 = c$. It is known from analytic geometry that such a set consists of the points in $\mathbb{R}^3$ which are orthogonal to θ and which lie a directed distance c from the origin. This set is therefore an ordinary plane and we see that definition 2.7 generalizes this concept to $\mathbb{R}^n$.

The equation $\langle \theta, x \rangle = 0$ is a rank 1 linear equation, so the set of points satisfying this equation forms an $n - 1$ dimensional vector space. Therefore, a hyperplane is just a parallel translate of such a vector space and therefore it also has dimension $n - 1$.

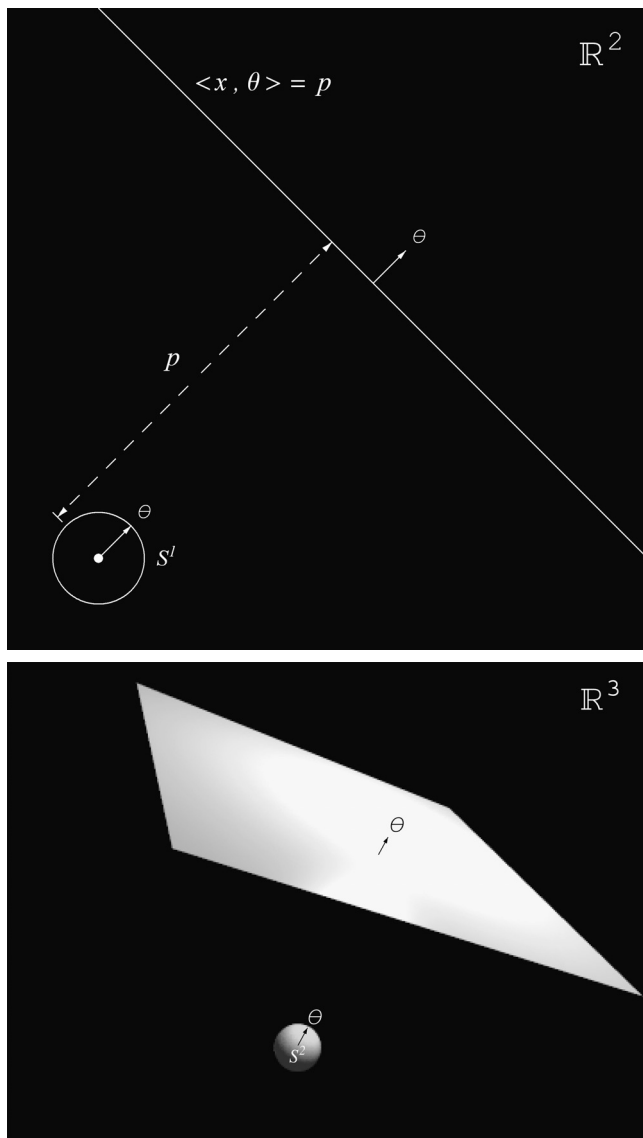
In $\mathbb{R}^2$ hyperplanes are lines, which are known to be one-dimensional. It is well known from calculus courses that equations of the form $\langle \theta, x \rangle = c$ in $\mathbb{R}^3$ determine planes that are two-dimensional. So in $\mathbb{R}^3$ hyperplanes are just ordinary planes. In $\mathbb{R}^n$ a hyperplane is $n - 1$ dimensional. As in $\mathbb{R}^2$ a hyperplane is uniquely determined by a unit vector θ and a signed distance c and we take the liberty of using the phrasing “the hyperplane $\langle \theta, x \rangle = c$ ” rather than the more precise but awkward expression “the hyperplane $\{x \in \mathbb{R}^n : \langle \theta, x \rangle = c\}$.” We take similar liberties with other sets, using the defining equation as a synonym for the set itself.

Because unit vectors uniquely determine directions we find that the concepts of “unit vector,” “direction,” and “point of the unit sphere” are identical. In particular, we can write “ $u \in S^{n-1}$ ” in lieu of “ u is a unit vector.” In the same way we can say “given a direction” in place of “given a unit vector.”

The plan now is to define the Radon transform in higher dimensions by using the integral $\int_L f(x) dx$ with the line L replaced by a hyperplane. Before doing this we need to investigate hyperplanes and integration on hyperplanes.

Here are illustrations of typical hyperplanes in $\mathbb{R}^2$ and in $\mathbb{R}^3$. Both hyperplanes are defined by the equation $\langle x, \theta \rangle = p$. The distance from the origin to the hyperplane is p ;

this is illustrated in the first figure which shows the hyperplane $\langle x, \theta \rangle = p$ in $\mathbb{R}^2$. This is of course the straight-line orthogonal to θ and a signed distance p from the origin. We have shown the unit sphere S^1 in $\mathbb{R}^2$, which is the unit circle in the figure. We have also shown the unit vector θ in two places: first, inside the sphere, and second, on the hyperplane. The second figure shows the analogous features in $\mathbb{R}^3$ but does not explicitly show the distance p . However, if you measured the distance from the origin to the plane in that diagram, the distance would be p .



Along with hyperplanes, the following concept is useful.

Definition 2.8. If $\theta \in S^{n-1}$, then $\theta^\perp$ denotes the linear space orthogonal to θ , namely,

$$\theta^\perp = \{x \in \mathbb{R}^n : \langle x, \theta \rangle = 0\}$$

Remark 2.9. It is clear that $\theta^\perp$ is a linear space of dimension $n - 1$ because it is defined by the rank 1 homogeneous linear equation $\langle x, \theta \rangle = 0$.

Remark 2.10. Occasionally we use the notation $\theta^\perp$ to represent a unit vector orthogonal to θ . The context will always resolve any ambiguity.

Remark 2.11. Recall that an *orthonormal basis* of a subspace V of $\mathbb{R}^n$ is a spanning set of unit vectors $\{\theta_1, \dots, \theta_d\}$ in V with the property that these unit vectors are mutually orthogonal.

There is another way of looking at hyperplanes. If $a \in \mathbb{R}^n$ and U is a subset of $\mathbb{R}^n$ we use the notation $U + a = a + U$ to denote the set formed by adding a to arbitrary elements of U . The set $U - a$ is defined similarly.

Proposition 2.12 (hyperplane lemma). *A subset H of $\mathbb{R}^n$ is a hyperplane orthogonal to a unit vector θ lying a distance s from the origin if and only if $H = \theta^\perp + s\theta$.*

If H is the hyperplane $\langle \theta, x \rangle = s$ and if $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ is an orthonormal basis of $\mathbb{R}^n$ then

$$H = \{x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta : x_1, \dots, x_{n-1} \in \mathbb{R}\} \quad (2.3)$$

Proof. By definition H is a hyperplane if and only if it is the set of points satisfying the equation $\langle \theta, x \rangle = s$ for some unit vector θ and some real number s . Define

$$V = H - s\theta \quad (2.4)$$

Because H is defined by the equation $\langle \theta, x \rangle = s$, we see that V is defined by the equation $\langle \theta, y \rangle = 0$. By Definition 2.8 this means that $V = \theta^\perp$ and equation (2.4) implies that $H = \theta^\perp + s\theta$. The proof of the converse, that any set of the form $\theta^\perp + s\theta$ is a hyperplane, is an easy variation of the preceding argument and is omitted.

If $H = \theta^\perp + s\theta$ then any element of H is of the form $u + s\theta$ where $u \in \theta^\perp$. Hence, if $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ is an orthonormal basis of $\mathbb{R}^n$ then it must be that $\{\theta_1, \theta_2, \dots, \theta_{n-1}\}$ is an orthonormal basis for $\theta^\perp$, so u is of the form $x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1}$ and the last statement follows immediately. ■

Remark 2.13. The symbol ■ is used to indicate the end of a proof.

2.2 Integration on Hyperplanes and the Definition of the Radon Transform

Almost all the basic theory of the Radon transform is a consequence of only two theorems from the calculus: the theorem on change of variables in an integral and the theorem on interchange of order of integration.

Before stating these theorems we need to develop some useful notation and to discuss the theory of integration.

We use the symbol

$$\int_{\mathbb{R}^n} f(x) dx$$

to denote the (multiple) integral of the function $f(x) = f(x_1, x_2, \dots, x_n)$ defined on $\mathbb{R}^n$. This avoids the awkward multiple integral notation:

$$\begin{aligned} & \iint f(x, y) dx dy \\ & \iiint f(u, v, w) du dv dw, \text{ and} \\ & \int \cdots \int f(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n \end{aligned}$$

and also allows one to see the many similarities in the theory of integration over different Euclidean spaces.

The type of integration studied in calculus courses is called **Riemann integration**. The theory of **Lebesgue integration** is an extension of Riemann integration in which any Riemann integrable function is also Lebesgue integrable and the Lebesgue and Riemann integrals have the same value. Therefore those readers who have studied the calculus already know many examples of Lebesgue integrable functions. Namely all the Riemann integrable functions studied in the calculus are also Lebesgue integrable, and there are many interesting Lebesgue integrable functions that are not Riemann integrable. Regardless, you can still read this chapter interpreting “integrable” to mean “Riemann integrable.” However, care has been taken to ensure that all results are also valid for Lebesgue integrable functions. An example of a Lebesgue integrable function is given in example 2.36, and even those readers with only a background in the calculus will be able to appreciate this example.

All the basic properties of Riemann integrals extend to Lebesgue integrals. One of the most important of these properties is linearity. Lebesgue integration is a linear operator in the sense that if f and g are Lebesgue integrable functions and c is a constant, then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) + g(x) dx &= \int_{\mathbb{R}^n} f(x) dx + \int_{\mathbb{R}^n} g(x) dx \\ \int_{\mathbb{R}^n} c \cdot f(x) dx &= c \cdot \int_{\mathbb{R}^n} f(x) dx \end{aligned}$$

Four features of Lebesgue integration are particularly important for us. First, it is built into the definition of the Lebesgue theory that a function f is Lebesgue integrable if and only if its absolute value, $|f|$, is Lebesgue integrable. This feature becomes crucial

in a few later arguments. For example, when we come to define the Fourier transform we need to integrate functions of the form $e^{-ix\omega} f(x)$ for real valued functions $f(x)$ and real constants ω . If f is known to be Lebesgue integrable, then so is $e^{-ix\omega} f(x)$ because these functions have the same absolute value.

The second important feature of Lebesgue integration is that it allows ∞ as a legal value of both functions and integrals. Functions defined on $\mathbb{R}^n$ fall into three mutually exclusive classes: those for which it is impossible to compute a Lebesgue integral, those for which the Lebesgue integral can be computed but has an infinite value, and those possessing finite Lebesgue integrals. A function in the latter class is called **Lebesgue integrable**, and the set of all Lebesgue integrable functions on $\mathbb{R}^n$ forms a vector space denoted by $L^1(\mathbb{R}^n)$. When we say that f is integrable or Lebesgue integrable we mean that not only can we compute $\int_{\mathbb{R}^n} f(x) dx$, but also that $\int_{\mathbb{R}^n} |f(x)| dx < \infty$. The same meaning is implied when we say $f \in L^1(\mathbb{R}^n)$. Sometimes we simplify by saying “integrable” rather than “Lebesgue integrable.”

The third important feature of Lebesgue integration is that it is fundamentally based on the notion of the measure of a set. We will not go into the general definition of the measure of a set. But we do note that if A is a set, then it may or may not have a measure. For those sets that possess measures we let the **measure** be denoted by the symbol $|A|$. Intuitively speaking, measure means length in $\mathbb{R}$, area in $\mathbb{R}^2$, and volume in $\mathbb{R}^3$. In $\mathbb{R}^n$ measure is the appropriate generalization of length, area, and volume to higher dimensions. However, we occasionally use the term “volume” as a synonym for measure, even in $\mathbb{R}^n$ with $n > 3$. Clearly, the measure of a set A depends on which Euclidean space contains A . For example the interval $[0, 1]$ has measure 1 as a subset of $\mathbb{R}$ and measure 0 as a subset of $\mathbb{R}^2$. In general, the context will determine which measure is being used. If it is necessary for clarity or emphasis we will say “the n -dimensional measure of A ,” otherwise we will just say “the measure of A .” Concerning the symbol $|A|$, there is no possible confusion with the norm or absolute value symbol because these do not apply to sets. The somewhat awkward phrasing “measure of a measurable set” is made necessary because there are examples of sets that are not measurable. Nonmeasurable sets, albeit plentiful in theory, are hard to come by practically and they will not play a role in this book. All the sets familiar to us in the calculus, such as intervals, graphs of continuous functions, etc., are measurable and their measures in the Lebesgue sense are exactly what we would expect from our experience with the calculus. For example, the measure of a rectangle in $\mathbb{R}^2$ is exactly the product of the lengths of the base and the altitude. A two dimensional rectangle has measure zero in $\mathbb{R}^3$, but the measure of the region contained inside a sphere of radius R in $\mathbb{R}^3$ has a measure exactly equal to its traditional volume: $\frac{4}{3}\pi R^3$.

Remark 2.14. If A is a measurable set, then we use the notation

$$\int_A f(x) dx$$

to denote the integral of f restricted to A . Sometimes we write

$$\int_{x \in A} f(x) dx$$

to emphasize the role of the variable of integration.

Sets of measure zero play a particularly useful role in the Lebesgue theory. Intuitively, a set of measure zero has a negligible amount of measure. A precise definition is possible, but all readers will either know the correct definition or else will be well served by this description.¹ One important thing about sets of measure zero is that if two Lebesgue integrable functions agree everywhere except on a set of measure zero, then they have the same integral. Therefore, sets of measure zero can be ignored as far as Lebesgue integration is concerned. Phrasing of the form “... agree except on a set of measure zero” occurs so often that we formalize the idea; we say that something happens *almost everywhere* if it happens except on a set of measure zero. We also say that a property is true for *almost all* x if it is true for all x except those in some set of measure zero.

Example 2.15. In $\mathbb{R}^2$, let f be the function which is identically 1 and let g be the function which is 1 except on the x_1 axis where it is zero. Then we can say that $f = g$ almost everywhere because the x_1 axis has measure zero in $\mathbb{R}^2$.

The fourth feature of the Lebesgue theory is that it allows an easy extension of the theory of integration to domains more general than $\mathbb{R}^n$. For example, there is a theory of Lebesgue integration on hyperplanes and on spheres. In the hyperplane and the sphere, Lebesgue integration is analogous to surface integration in Euclidean 3-space. Also, in the sphere, it is traditional to speak of the volume of a sphere when we really mean the measure. Of course the sphere, S^{n-1} , has n -dimensional measure zero as a subset of $\mathbb{R}^n$. But when we consider S^{n-1} as a space by itself and compute its volume via its $(n-1)$ dimensional Lebesgue measure, then we get a positive result. Before stating the formula for computing the volume of a sphere, we introduce the concept of a ball in $\mathbb{R}^n$. The open *unit ball* of $\mathbb{R}^n$ is denoted by the symbol B^n , or just B if the dimension is understood. It is defined by the inequality $|x| < 1$. The ball B^2 is a disk and the ball B^3 is the interior of the sphere S^2 . In general, the unit sphere S^{n-1} is the boundary of the unit ball B^n . Occasionally people speak of the volume of the sphere when they actually mean the volume of the ball. This is clarified by noting that $|S^{n-1}|$ refers to the $(n-1)$ -dimensional measure of the unit sphere and that this is different than the n -dimensional volume of the ball. We frequently will write B in place of B^n , if the dimension is known. However, we will always write S^{n-1} for the unit sphere.

¹ Okay, so here is the precise definition. First, a rectangle in $\mathbb{R}^n$ is a generalization of a rectangle in $\mathbb{R}^2$; it is a set of the form $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. We define $\mu(R) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$ and call this the (n) -dimensional measure of the rectangle. Then we say that a set B has *Lebesgue measure* 0. If for every $\varepsilon > 0$ we can find a sequence of rectangles $R_1, R_2, \dots, R_k, \dots$ such that B is contained in the union of these rectangles and $\sum_{k=1}^{\infty} \mu(R_k) < \varepsilon$.

The following theorem details the volumes of spheres and balls in n -dimensional space. The proofs of these results may be found in Flanders [176] or Stroock [588].

Theorem 2.16.

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (2.5)$$

$$\begin{aligned} |B^n| &= \frac{1}{n} |S^{n-1}| \\ &= \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})} \end{aligned}$$

where Γ denotes the usual gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

defined for $x > 0$.²

The fact that $\frac{1}{n}|S^{n-1}| = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}$ is an easy consequence of the elementary properties of the gamma function.

We can easily compute the volumes of some spheres:

$$\begin{aligned} |S^1| &= 2\pi \quad |S^3| = 2\pi^2 \\ |S^2| &= 4\pi \quad |S^4| = \frac{8}{3}\pi^2 \end{aligned}$$

The first two of these formulas should be very familiar: they give the circumference of a unit circle in $\mathbb{R}^2$ and the area of a unit sphere in $\mathbb{R}^3$. The last equation is the 4-dimensional volume of the 4-sphere in $\mathbb{R}^5$.

If $S^{n-1}(c, R)$ denotes the sphere with center c and radius R in $\mathbb{R}^n$, and if $B^n(c, R)$ denotes the open ball with center c and radius R in $\mathbb{R}^n$, then

$$\begin{aligned} |S^{n-1}(c, R)| &= |S^{n-1}| R^{n-1} \\ |B^n(c, R)| &= |B^n| R^n \end{aligned}$$

For example, in $\mathbb{R}^3$ we get the familiar formulas $4\pi R^2$ for the area of the sphere and $\frac{4}{3}\pi R^3$ for the volume of a ball.

At this point we have given enough of an overview of the Lebesgue theory so that all readers can understand the statements and proofs that are developed in this chapter. From this point on we assume that the term **integrable** means Lebesgue integrable. This assumption should work even for readers who have never met a function that is Lebesgue integrable but not Riemann integrable. The only difference is that when those

² A fundamental identity for the gamma function is $\Gamma(x+1) = x\Gamma(x)$ from which we easily derive $\Gamma(n) = (n-1)!$ when n is a natural number. It can be shown that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. From these facts we can find $\Gamma(\frac{n}{2})$ for any positive integer n .

readers picture our results, they will picture the Riemann integrable functions that they studied in the calculus. This includes all continuous functions of bounded support. It also includes the so-called “pixel” functions, which are functions defined on subsquares of a fixed square and which are constant on each subsquare. These functions are useful in image processing.

So, what is the advantage of using the Lebesgue theory? First, most proofs become easier, because all the hard work has been done in developing the Lebesgue theory. Also, because many discontinuous functions arise in tomography, it is advantageous to use the Lebesgue theory, which handles these issues very smoothly. For example, it is quite possible to have a very nice Riemann integrable function f which is continuous almost everywhere on $\mathbb{R}^n$. In tomography we would like to integrate this function over all hyperplanes. But it is also quite possible that on some of these hyperplanes f will have discontinuities of positive measure and thus not be Riemann integrable. Also it is hard using just the Riemann theory to determine how many of these hyperplanes there are. On the other hand, it is a very simple matter to deal with this by using the Lebesgue theory.

Many good references are available for those readers interested in learning or reviewing the Lebesgue theory of integration, but we will mention only five: Hewitt and Stromberg [304], Strichartz [587], Evans and Gariepy [153], Kuttler [375], and Stroock [588]. The Stroock reference is a particularly focused introduction to the Lebesgue theory. You will find the theorem on change of variable and Fubini’s theorem in these references.

Having finished this short tour of the Lebesgue theory of integration we can now precisely state the theorem on change of variables and Fubini’s theorem on iterated integration.

Theorem 2.17 (Change of Variables in Integration).³ *If g is a one-to-one, continuously differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$, if f is an integrable function on $\mathbb{R}^n$, and if U is an open set in $\mathbb{R}^n$, then*

$$\int_{g(U)} f(x) dx = \int_U f(g(y)) Jg(y) dy$$

The function Jg is the Jacobian determinant defined as follows.

Definition 2.18. Let g be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ which possesses first partial derivatives of all orders. Define the matrix

$$g'(x) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

³ This can also be called the theorem on substitution. It gives the effect of the substitution $x = g(y)$ on the integral $\int_{g(U)} f(x) dx$.

and define the **Jacobian** of g to be the function

$$Jg(x) = |\det(g'(x))|$$

The theorem on interchange of order of integration is called Fubini's theorem and the following convention is useful there and in several other instances later on: if x is a variable element of $\mathbb{R}^n$ then we define

$$\begin{aligned} x' &= (x_1, \dots, x_k) \text{ and} \\ x'' &= (x_{k+1}, \dots, x_n) \end{aligned}$$

Therefore, we can write $x = (x', x'')$ and, in the same way, a function f of n variables can be written $f(x', x'')$. When we encounter this notation we must keep in mind that $x' \in \mathbb{R}^k$ actually represents a k -tuple of numbers and $x'' \in \mathbb{R}^{n-k}$ represents an $n-k$ -tuple, so (x', x'') represents an n -tuple of numbers, a typical element of $\mathbb{R}^n$. The context will generally clarify what the correct value of k is. However, when we use the symbol x' in connection with a variable x without mention of k , then we will always assume that $k = n-1$, that is, that $x' = (x_1, \dots, x_{n-1})$. In the same way, the differential $dx' = dx_1 \cdots dx_{n-1}$ denotes the volume element on $\mathbb{R}^{n-1}$.

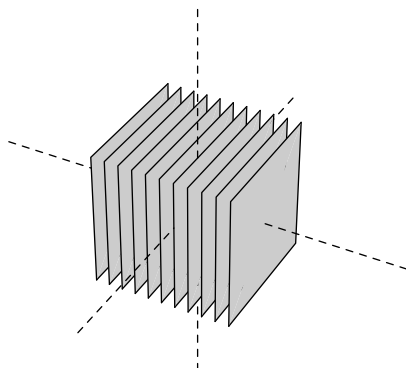
Theorem 2.19 (Fubini's theorem on changing the order of integration). *If f is an integrable function on $\mathbb{R}^n$, then the function assigning $x'' \in \mathbb{R}^{n-k}$ to $\int_{x' \in \mathbb{R}^k} f(x', x'') dx'$ is integrable. Hence $\int_{x' \in \mathbb{R}^k} f(x', x'') dx'$ exists for almost all x'' in $\mathbb{R}^{n-k}$. Furthermore*

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) dx &= \int_{x'' \in \mathbb{R}^{n-k}} \left(\int_{x' \in \mathbb{R}^k} f(x', x'') dx' \right) dx'' \\ &= \int_{x' \in \mathbb{R}^k} \left(\int_{x'' \in \mathbb{R}^{n-k}} f(x', x'') dx'' \right) dx' \end{aligned}$$

Conversely, if the function assigning x'' to $\int_{x' \in \mathbb{R}^k} |f(x', x'')| dx'$ is integrable on $\mathbb{R}^{n-k}$, then f is integrable on $\mathbb{R}^n$ and the previous equation holds.

Except for the notational convention on multiple integrals and the terminology about a function existing "almost everywhere," the theorem on change of variable in an integral and Fubini's theorem are close enough to the corresponding theorems in the calculus to make the validity of the theorems plausible. The interested reader can find the proofs in the references listed above theorem 2.17.

Remark 2.20. Let f be integrable on $\mathbb{R}^n$. In the following figure we see $\mathbb{R}^n$ sliced into hyperplanes $\langle \theta, x \rangle = p$. These hyperplanes are parametrized by p and we can compute the integral of f over each such hyperplane to obtain a function $I(p)$ of the single variable p . The geometric meaning of Fubini's theorem is that $I(p)$ is integrable over the real line and its integral is precisely the multiple integral $\int_{\mathbb{R}^n} f(x) dx$.



Fubini's theorem: dissection of $\mathbb{R}^n$ into hyperplanes.

Remark 2.21. In the Lebesgue theory it is quite possible to have an integrable function that is finite everywhere, yet which is not integrable on some hyperplanes. A consequence of Fubini's theorem is that there cannot be too many such hyperplanes where the integral is infinite, compare proposition 2.30 and example 2.36.

An important special case of the change of variables theorem occurs when g is a linear transformation with determinant ± 1 and where the open set U is all of $\mathbb{R}^n$. Then by linear algebra we obtain that $g(U) = \mathbb{R}^n$ also. If we let A denote the linear transformation and apply the theorem with $g = A$ we then get

Corollary 2.22. *A linear transformation with determinant ± 1 leaves integrals invariant:*

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f(A(y)) dy.$$

The **translation** of a function f by a vector a is the function whose value at x is $f(x - a)$ and is denoted by f_a . A translation is of the form $f \circ T$ where $T(x) = x - a$. It is obvious that the Jacobian of T is identically equal to 1 and we immediately have the following important result:

Corollary 2.23 (Invariance of integration under translations). *Integration is invariant under translations: if a is a fixed point in $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f_a(x) dx = \int_{\mathbb{R}^n} f(x - a) dx.$$

A matrix is said to be orthogonal if $AA^t = I$. This happens if and only if the columns of A form an orthonormal basis of $\mathbb{R}^n$. A linear transformation is said to be an orthogonal transformation if it preserves lengths, that is, if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$. It is easily shown that a linear transformation is orthogonal if and only if its associated matrix is orthogonal. Orthogonal matrices give rise to linear transformations

with determinant ± 1 . This easy observation from the defining equation is the basis of the next corollary.

Corollary 2.24. *If $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f(y_1\theta_1 + y_2\theta_2 \cdots + y_{n-1}\theta_{n-1} + y_n\theta_n) dy \quad (2.6)$$

Proof. If $B = \{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then its matrix A is orthogonal, so by the previous corollary

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f(A(y)) dy$$

If we now let $y = (y_1, y_2, \dots, y_{n-1}, y_n)$ be an arbitrary element of $\mathbb{R}^n$, then $A(y) = y_1\theta_1 + y_2\theta_2 \cdots + y_{n-1}\theta_{n-1} + y_n\theta_n$ and hence we see that equation (2.6) is correct. ■

Remark 2.25. This corollary is the first instance of a useful general principle. Sometimes it is helpful to re-express the argument $x = (x_1, x_2, \dots, x_n)$ in a more natural coordinate system. If $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta_n\}$ is an orthonormal basis of $\mathbb{R}^n$, then the expression

$$x = y_1\theta_1 + y_2\theta_2 \cdots + y_{n-1}\theta_{n-1} + y_n\theta_n$$

is just as valid as

$$x = (x_1, x_2, \dots, x_n)$$

In this case we can consider the values $y_1, y_2, \dots, y_n$ to be new coordinates for x . In a sense the symbols “ $\theta_j +$ ” play the same role as the commas “,” do in the standard coordinate system.

An important consequence of these theorems is the technique of using polar coordinates in integrals. Most readers will be familiar with the change of variable to polar coordinates in the plane: if f is integrable on the plane, then

$$\int_{\mathbb{R}^2} f(x) dx = \int_0^{2\pi} \int_0^\infty f(r(\cos \alpha, \sin \alpha)) r dr d\alpha \quad (2.7)$$

and conversely, if the integral on the right exists, then f is integrable on the plane. The vector $(\cos \alpha, \sin \alpha)$ is just a typical unit vector, so the outer integration is actually integration over the circle S^1 . So using $\theta = (\cos \alpha, \sin \alpha)$ we can rewrite the previous equation as

$$\int_{\mathbb{R}^2} f(x) dx = \int_{S^1} \int_0^\infty f(r\theta) r dr d\theta$$

We will not go into the details, but theorem 2.17, on change of variables, and Theorem 2.19, Fubini's theorem, validate the following generalization of the polar coordinate formula, equation (2.7), to $\mathbb{R}^n$.

$$\int_{\mathbb{R}^n} f(x) dx = \int_{S^{n-1}} \int_0^\infty f(r\theta) r^{n-1} dr d\theta \quad (2.8)$$

Note that in going to higher dimensions the sphere becomes S^{n-1} whereas the power of r changes to r^{n-1} . A special case occurs when f is a function only of $|x|$ in which case the function is constant with respect to θ . The integration over the sphere therefore gives the volume of the sphere, which is denoted by $|S^{n-1}|$. The formula then becomes

$$\begin{aligned} \int_{\mathbb{R}^n} f(|x|) dx &= |S^{n-1}| \int_0^\infty f(r) r^{n-1} dr \\ &= \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty f(r) r^{n-1} dr \end{aligned} \quad (2.9)$$

The books by Strichartz [587] and Stroock [588] contain derivations of the polar coordinate substitution in n -dimensional Euclidean space.

According to the plan for generalizing the Radon transform, we need to be able to integrate functions f defined on $\mathbb{R}^n$ over hyperplanes $\langle \theta, x \rangle = s$. We therefore introduce the following definition.

Definition 2.26 (Integration over hyperplanes). Given a hyperplane H defined by $\langle \theta, x \rangle = s$ and a function f defined on $\mathbb{R}^n$, let $\{\theta_1, \theta_2, \dots, \theta_{n-1}\}$ be an orthonormal basis of $\theta^\perp$. Then, letting $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, we define the **hyperplane integral of f over H** by

$$\int_{\langle \theta, x \rangle = s} f(x) dx = \int_{x' \in \mathbb{R}^{n-1}} f(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta) dx'$$

Remark 2.27. If $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ is an orthonormal basis of $\mathbb{R}^n$, then it is immediate that $\{\theta_1, \theta_2, \dots, \theta_{n-1}\}$ is an orthonormal basis of $\theta^\perp$. The hyperplane lemma then shows that the argument in the integrand is a typical element of the hyperplane $\langle \theta, x \rangle = s$. This justifies the definition. We will frequently make use of an orthonormal basis of $\mathbb{R}^n$ whose last vector is θ . It will go without saying, from now on, that the first $n - 1$ vectors in such a basis are an orthonormal basis of $\theta^\perp$.

We are now in a position to generalize the preliminary definition of the Radon transform given earlier.

Definition 2.28 (The Radon transform). The **Radon transform** is an operator R defined on $L^1(\mathbb{R}^n)$ whereby for any integrable function f on $\mathbb{R}^n$, the function Rf is

defined for $\theta \in S^{n-1}$ and $s \in \mathbb{R}$ by

$$Rf(\theta, s) = \int_{\langle \theta, x \rangle = s} f(x) dx$$

whenever the integral exists. The left-hand side is read “the value of the Radon transform of the function f on the hyperplane $\langle \theta, x \rangle = s$.” An alternate expression is

$$Rf(\theta, s) = \int_{y \in \theta^\perp} f(y + s\theta) dy$$

where this integral is just a convenient expression for the hyperplane integral

$$\int_{x' \in \mathbb{R}^{n-1}} f(y_1\theta_1 + y_2\theta_2 \dots + y_{n-1}\theta_{n-1} + s\theta) dx'$$

($\{\theta_1, \theta_2, \dots, \theta_{n-1}\}$ being an orthonormal basis for $\theta^\perp$).

For a fixed $\theta \in S^{n-1}$ we also define the **Radon projection (orthogonal to θ)**, $R_\theta f$, by

$$R_\theta f(s) = Rf(\theta, s)$$

Remark 2.29. The domain of the Radon transform of a specific function is $S^{n-1} \times \mathbb{R}$, because any element of $S^{n-1} \times \mathbb{R}$ is of the form (θ, s) with $\theta \in S^{n-1}$ and $s \in \mathbb{R}$. We define the symbol $Z^n = S^{n-1} \times \mathbb{R}$ and call it the **cylinder** (of dimension n). A moment’s visualization of the product $S^1 \times \mathbb{R}$ justifies the name. The reason for the symbol Z is that “cylinder” is spelled “zyylinder” in German. For those readers who know some topology we note that Helgason [291] lets the domain be P_n , the set of all hyperplanes in $\mathbb{R}^n$. The space P_n is homeomorphic to real projective space $\mathbb{R} \mathbb{P}^n$ and there is a natural double covering of P_n by Z^n .

The following questions now arise.

1. Under what conditions will the integral in Definition 2.26 exist? This is the same as asking “when does $Rf(\theta, s)$ exist?”
2. If we choose a different basis, will we arrive at the same answer for the hyperplane integral?
3. The hyperplane defined by $\langle \theta, x \rangle = s$ is the same as the one defined by $\langle -\theta, x \rangle = -s$. Will the integrals be the same? This is the same as asking if the Radon transform is an even function on the cylinder.

These questions are answered in the next few results.

Proposition 2.30 (The hyperplane integration theorem). *Let $\theta \in S^{n-1}$. If $f \in L^1(\mathbb{R}^n)$, then $\int_{\langle \theta, x \rangle = s} f(x) dx$ exists for almost all $s \in \mathbb{R}$ and is independent of the*

orthonormal basis used to define the hyperplane. Furthermore

$$\int_{s=-\infty}^{\infty} \int_{\langle \theta, x \rangle = s} f(x) dx ds = \int_{\mathbb{R}^n} f(w) dw \quad (2.10)$$

Proof. Let $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ be an orthonormal basis of $\mathbb{R}^n$. The linear transformation that changes the coordinates of $\mathbb{R}^n$ from one orthonormal basis to another is orthogonal. Applying this idea to the standard basis and the given basis, Corollary 2.24, shows that

$$\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} f(y_1\theta_1 + y_2\theta_2 \cdots + y_{n-1}\theta_{n-1} + y_n\theta) dy$$

Now apply Fubini's theorem to the integral on the right, in the y -coordinate system. This gives two results. First, that for almost every s ,

$$\int_{y' \in \mathbb{R}^{n-1}} f(y_1\theta_1 + y_2\theta_2 \cdots + y_{n-1}\theta_{n-1} + s\theta) dy'$$

is integrable, which proves the statement that $\int_{\langle \theta, x \rangle = s} f(x) dx$ exists for almost all $s \in \mathbb{R}$, and, second, that equation (2.10) is valid.

To show the independence of the orthonormal basis used in definition 2.26 let $A = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$ and $B = \{\beta_1, \beta_2, \dots, \beta_{n-1}\}$ be orthonormal bases of $\theta^\perp$ and let C be the $(n-1) \times (n-1)$ matrix which is the matrix of the change of basis from A to B . Then it is easy to show that C is an orthogonal matrix and that if $v = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + s\theta$ is an element of the hyperplane $\langle \theta, x \rangle = s$, then we also have $v = y_1\beta_1 + y_2\beta_2 + \cdots + y_{n-1}\beta_{n-1} + s\theta$ where

$$y' = Cx' \quad (2.11)$$

(as usual $y' = (y_1, \dots, y_{n-1})$, $x' = (x_1, \dots, x_{n-1})$). If we apply the change of variable theorem on $\mathbb{R}^{n-1}$ to the integral

$$\int_{\langle \theta, x \rangle = s} f(x) dx = \int_{x' \in \mathbb{R}^{n-1}} f(x_1\alpha_1 + x_2\alpha_2 + \cdots + x_{n-1}\alpha_{n-1} + s\theta) dx'$$

with the change of variables (2.11) then we also get

$$\int_{\langle \theta, x \rangle = s} f(x) dx = |\det(C)| \int_{y' \in \mathbb{R}^{n-1}} f(y_1\beta_1 + y_2\beta_2 \cdots + y_{n-1}\beta_{n-1} + s\theta) dy'$$

But C is orthogonal, so $|\det(C)| = 1$ and this proves the hyperplane integral is independent of the orthonormal basis used in the definition. ■

Corollary 2.31. $\int_{\langle \theta, x \rangle = s} f(x) dx = \int_{\langle -\theta, x \rangle = -s} f(x) dx$

The proof is really easy; just apply the definitions.

Corollary 2.32 (The Radon transform is even). *The Radon transform has the following symmetry property.*

$$Rf(-\theta, -s) = Rf(\theta, s)$$

Proof. This is a direct consequence of the definition of the Radon transform and the previous corollary. ■

The next corollary tells us that the values of the Radon projection of an integrable function exist as finite real numbers for almost all, if not all, s . It is an immediate consequence of the hyperplane integration theorem and its proof is omitted.

Corollary 2.33. *If f is an integrable function on $\mathbb{R}^n$ and if $\theta \in S^{n-1}$ is a fixed direction, then $R_\theta f(s)$ is defined for almost all s in $\mathbb{R}$. Furthermore*

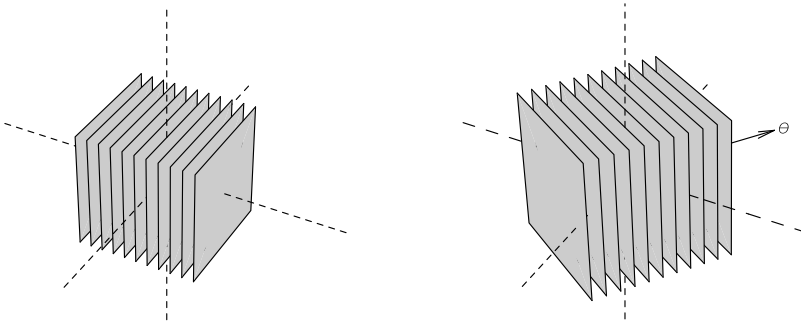
$$\int_{s \in \mathbb{R}} R_\theta f(s) ds = \int_{x \in \mathbb{R}^n} f(x) dx$$

This also shows that the projection of an integrable function is again integrable.

The next corollary is also an immediate consequence of the hyperplane integration theorem.

Corollary 2.34. *The integral defining $Rf(\theta, s)$ is independent of the particular orthonormal basis used in the definition of the Radon transform.*

Remark 2.35. The next two figures give an intuitive idea of Proposition 2.30 and the interplay between Fubini's theorem and the change of variables theorem. The first figure illustrates Fubini's theorem, showing the domain of f sliced into hyperplanes parallel to the usual coordinate axes. The second picture shows the same set of hyperplanes rotated to be orthogonal to a unit vector θ . The essence of the proposition is that f will be integrable on almost every hyperplane and that the value of $\int_{x \in \mathbb{R}^n} f(x) dx$ is independent of this rotation.



So if f is integrable on $\mathbb{R}^n$, then its restriction to almost every hyperplane is integrable. However, as the next example shows, this does not imply that f is integrable on *every* hyperplane.

Example 2.36. Example of an integrable function f on $\mathbb{R}^n$ which is finite everywhere, yet for which there exists a hyperplane on which f is not integrable.

Construction: let

$f(x) = (x_1^2 + \cdots + x_{n-1}^2)^{-\frac{n-1-\sqrt{|x_n|}}{2}}$ if $(x_1^2 + \cdots + x_{n-1}^2) \leq 1$ and $0 < |x_n| \leq 1$ with f defined to be zero elsewhere.

Let $\theta = (0, 0, \dots, 0, 1)$. Note that by the definition of θ the hyperplane $\langle \theta, x \rangle = s$ simplifies to $x_n = s$ and we can apply the hyperplane integration theorem to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) dx &= \int_{-\infty}^{\infty} \int_{\langle \theta, x \rangle = s} f(x) dx ds \\ &= \int_{-1}^1 \int_{|x'| \leq 1} (x_1^2 + \cdots + x_{n-1}^2)^{-\frac{n-1-\sqrt{|s|}}{2}} dx' ds \end{aligned} \quad (2.12)$$

The restriction of the x' variable to the unit ball in $\mathbb{R}^{n-1}$ is justified by the definition of the function f .

Focus attention on the inner integral. This integration is being performed over $\mathbb{R}^{n-1}$. Use the polar coordinate substitution (2.9), but of course applied with $n-1$ in place of n . We then have

$$x' = r'\theta'$$

where

$$r' = \sqrt{x_1^2 + \cdots + x_{n-1}^2}$$

and where $\theta' \in S^{n-2}$, which is the unit sphere of $\mathbb{R}^{n-1}$. Therefore, when we use the formula for polar coordinates we must use S^{n-2} and the power r'^{n-2} . This yields

$$\begin{aligned} \int_{\langle \theta, x \rangle = s} f(x) dx &= |S^{n-2}| \int_0^1 r'^{-(n-1-\sqrt{|s|})} r'^{n-2} dr' \\ &= |S^{n-2}| \frac{1}{\sqrt{|s|}} \end{aligned} \quad (2.13)$$

for $|s| \leq 1$. Now if $s = 0$ the last expression is infinite, whereas if $0 < |s| \leq 1$, it is finite. Because f is 0 for larger values of $|s|$ the hyperplane integral is also 0 and hence finite. Therefore f is integrable on almost every but not every hyperplane. However, f is integrable because applying equation (2.13) to the iterated integral in equation (2.12) gives

$$\int_{\mathbb{R}^n} f(x) dx = \int_{-1}^1 |S^{n-2}| \frac{1}{\sqrt{|s|}} ds = 4 |S^{n-2}| < \infty.$$

Remark 2.37. Hence, there are integrable functions f for which there is a direction θ and a point p such that $Rf(\theta, p)$ does not exist. Nonetheless, it is possible to show that $Rf(\theta, p)$ does exist almost everywhere when f is an integrable function. Proposition 2.30 was a result in that direction and the full result is proved in Chapter 3, Theorem 3.35. In fact it is not necessary for f to be integrable for $Rf(\theta, p)$ to exist as a finite Lebesgue integral almost everywhere. Theorem 3.35 shows that one only needs to have the integrability of $(1 + |x|)^{-1} f(x)$ for $Rf(\theta, p)$ to exist almost everywhere, and Theorem 3.36 shows that this is a necessary and sufficient condition when f is nonnegative.

At this point the following question may have occurred to you: Why do we integrate over hyperplanes rather than lines in the definition of the Radon transform?

Indeed lines generalize to any linear space, so we could have decided to integrate over lines instead of hyperplanes. In chapter 3 we define what is called the ***k-plane transform***. This takes a function on $\mathbb{R}^n$ and integrates over translates of k -dimensional subspaces. If $k = n - 1$, then this is precisely the Radon transform as we defined it. If $k = 1$ this integrates over lines and is called the ***x-ray transform***. In dimension 2, there is no practical difference between the Radon transform and the x-ray transform, whereas in higher dimensions there are significant differences.

We have chosen the term Radon transform for the operator that integrates a function over hyperplanes for reasons both of tradition and expediency.

The field of mathematics called integral geometry is closely related to mathematical tomography. One aspect of integral geometry is the study of properties of manifolds that can be determined from integration over submanifolds. Clearly, the Radon transform fits neatly into this aspect of integral geometry. The reader may refer to the books by Gardner [185], Nachbin [433], and Santaló [543] for an introduction to the ideas of integral geometry. Santaló's book presents a more classical form of integral geometry. Ren [515] also treats this more classical form of integral geometry. A typical question in this field is Buffon's needle problem: If one randomly drops a needle of length N on a ruled sheet of paper, where the lines have distance D from each other and $N \leq D$, then what is the probability of the needle touching a line? The answer is $\frac{2N}{\pi D}$.

2.3 Properties of the Radon Transform

All functions in this section are assumed to be integrable on $\mathbb{R}^n$.

The first two results are identical with corollaries 2.33 and 2.32 of the hyperplane integration theorem (Proposition 2.30) but are repeated here for convenience.

Proposition 2.38 (Existence of Radon projections). *If f is an integrable function on $\mathbb{R}^n$ and if $\theta \in S^{n-1}$ is a fixed direction, then $R_\theta f(s)$ is defined for almost all s in $\mathbb{R}$. Furthermore*

$$\int_{s \in \mathbb{R}} R_\theta f(s) ds = \int_{x \in \mathbb{R}^n} f(x) dx.$$

Proposition 2.39 (Even symmetry of the Radon transform). *The Radon transform has the following symmetry property:*

$$Rf(-\theta, -s) = Rf(\theta, s)$$

This is to say that Rf is an even function on $S^{n-1} \times \mathbb{R}$.

Because integration is a linear operator, it follows that the Radon transform is also a linear operator. Therefore, we immediately have Proposition 2.40.

Proposition 2.40 (Linearity of the Radon transform). *For any functions $f, g \in L^1(\mathbb{R}^n)$ and any constants a, b we have*

$$R(af + bg)(\theta, s) = aRf(\theta, s) + bRg(\theta, s)$$

Two important transformations of Euclidean space are translations and orthogonal linear transformations. The next two results show the effect of these on the Radon transform. First, the reader should recall the definition of composition of functions: $f \circ g(x) = f(g(x))$. We recall that the translation of a function f by a vector a is the function whose value at x is $f(x - a)$ and is denoted by f_a . Because a translation shifts the graph of a function in the direction a the next result is sometimes called the *shift theorem* for the Radon transform.

Proposition 2.41 (Translation-Shift Theorem).

$$Rf_a(\theta, s) = Rf(\theta, s - \langle \theta, a \rangle)$$

In words, the Radon projection of the translation of f by a is the translation of the Radon projection of f by $\langle \theta, a \rangle$.

Proof. If $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ is an orthonormal basis of $\mathbb{R}^n$, then

$$\begin{aligned} Rf_a(\theta, s) &= \int_{x' \in \mathbb{R}^{n-1}} f_a(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta) dx' \quad (2.14) \\ &= \int_{x' \in \mathbb{R}^{n-1}} f(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta - a) dx' \end{aligned}$$

By the orthonormality of the basis we can expand the vector a in the following way: define $b_k = \langle a, \theta_k \rangle$ for $k = 1, \dots, n-1$. Then,

$$a = \sum_{k=1}^{n-1} b_k \theta_k + \langle a, \theta \rangle \theta$$

so we can rewrite equation (2.14) as

$$\begin{aligned} &Rf_a(\theta, s) \\ &= \int_{x' \in \mathbb{R}^{n-1}} f((x_1 - b_1)\theta_1 + \dots + (x_{n-1} - b_{n-1})\theta_{n-1} + (s - \langle a, \theta \rangle)\theta) dx'. \end{aligned}$$

Because of the translation invariance of integration on Euclidean spaces, in this case on $\mathbb{R}^{n-1}$, we can treat $(x_j - b_j)$ as variables. Hence the last expression is the hyperplane integral of f on the hyperplane $\langle x, \theta \rangle = s - \langle a, \theta \rangle$. This is the same as $Rf(\theta, s - \langle a, \theta \rangle)$ and the proof is finished. ■

Proposition 2.42 (Rotation theorem). *Let T be an orthogonal transformation of $\mathbb{R}^n$. Then*

$$R(f \circ T)(\theta, s) = Rf(T(\theta), s)$$

Proof. If $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ is an orthonormal basis of $\mathbb{R}^n$, then

$$\begin{aligned} R(f \circ T)(\theta, s) &= \int_{x' \in \mathbb{R}^{n-1}} (f \circ T)(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta) dx' \\ &= \int_{x' \in \mathbb{R}^{n-1}} f(x_1T(\theta_1) + x_2T(\theta_2) + \dots + x_{n-1}T(\theta_{n-1}) + sT(\theta)) dx'. \end{aligned}$$

We used the linearity of T to expand the linear combination in this equation. But the orthogonality of the transformation T implies that $\{T(\theta_1), T(\theta_2), \dots, T(\theta_{n-1}), T(\theta)\}$ is an orthonormal basis of $\mathbb{R}^n$, so by the definition of hyperplane integration, the last expression is precisely $\int_{\langle T(\theta), x \rangle = s} f(x) dx = Rf(T(\theta), s)$. ■

A **radial function** is one of the form $f(|x|)$, where f is a function of a single real variable. Radial functions are precisely those that are invariant under all rotations. Therefore, the next result is an immediate consequence of the rotation theorem.

Corollary 2.43 (Radial function theorem). *If f is a radial function, then its Radon transform is direction independent. More precisely: if $f(x) = g(|x|)$ and f is integrable on $\mathbb{R}^n$, then for any directions $\theta, \eta \in S^{n-1}$ we have*

$$Rf(\theta, p) = Rf(\eta, p)$$

for almost all p .

Note that $f = g(|x|)$ is integrable on $\mathbb{R}^n$ if and only if $g(r)r^{n-1}$ is integrable on the real line, as the polar coordinate theorem shows.

Remark 2.44. The following precise integral formula for the Radon transform of a radial function $f(x) = g(|x|)$ is a special case of theorem 3.31, Chapter 3.

$$Rf(\theta, p) = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_p^\infty g(r) (r^2 - p^2)^{\frac{n-3}{2}} r dr.$$

The rotation theorem can be generalized to arbitrary nonsingular linear transformations but requires some more machinery. First, recall the fundamental theorem of

calculus: if φ is continuous on $\mathbb{R}$ and a is any real number, then

$$\frac{d}{dx} \int_a^x \varphi(t) dt = \varphi(x).$$

Writing the left-hand side in terms of limits leads to

$$\lim_{h \rightarrow 0} \frac{\int_a^{x+h} \varphi(t) dt - \int_a^x \varphi(t) dt}{h} = \lim_{h \rightarrow 0} \frac{\int_x^{x+h} \varphi(t) dt}{h}$$

so we end up with

$$\varphi(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} \varphi(t) dt$$

for any continuous function φ . A fundamental result in the Lebesgue theory is that this equation is true almost everywhere for integrable functions. Precisely we have lemma 2.45.

Lemma 2.45 (Lebesgue's differentiation theorem). *If φ is integrable on $\mathbb{R}$, then*

$$\varphi(x) = \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} \varphi(t) dt$$

almost everywhere.

The proof of this lemma may be found in various sources, for example, Kuttler [375], Corollary 14.9, or Stein [581], corollary I.1.

Next, we define a transform auxiliary to the Radon transform.

Definition 2.46. If f is integrable on $\mathbb{R}^n$ define

$$Mf(\theta, s) = \int_{\langle \theta, x \rangle \leq s} f(x) dx$$

Lemma 2.47. *Let $\theta \in S^{n-1}$, let $s \in \mathbb{R}$ and let f be integrable on $\mathbb{R}^n$. Then*

$$\frac{\partial}{\partial s} Mf(\theta, s) = Rf(\theta, s)$$

Proof. Because θ is a unit vector it is easy to see that an orthogonal change of variables will reduce us to the case where θ is the standard unit vector $(0, \dots, 0, 1)$. In this case $\langle \theta, x \rangle = x_n$ and the domain of integration for $Mf(\theta, s)$ is $\{x_n \leq s\}$. The integral defining $Mf(\theta, s)$ can be calculated by Fubini's theorem as follows. First, define

$$\varphi(x_n) = \int_{x' \in \mathbb{R}^{n-1}} f(x', x_n) dx'$$

Then φ represents the inner integral in the application of Fubini's theorem:

$$\begin{aligned} Mf(\theta, s) &= \int_{x_n \leq s} f(x) dx \\ &= \int_{-\infty}^s \int_{x' \in \mathbb{R}^{n-1}} f(x', x_n) dx' dx_n \\ &= \int_{-\infty}^s \varphi(x_n) dx_n \end{aligned}$$

Furthermore, Fubini's theorem guarantees that φ is integrable on the real line.

Therefore,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{Mf(\theta, s+h) - Mf(\theta, s)}{h} &= \lim_{h \rightarrow 0} \frac{\int_{-\infty}^{s+h} \varphi(x_n) dx_n - \int_{-\infty}^s \varphi(x_n) dx_n}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} \varphi(x_n) dx_n \\ &= \varphi(s) \end{aligned}$$

almost everywhere, the last step being justified by the integrability of φ and Lebesgue's differentiation theorem. This establishes the existence of the partial derivative and the equation

$$\frac{\partial}{\partial s} Mf(\theta, s) = \varphi(s)$$

The proof is completed by the easy observation that $\varphi(s) = \int_{\langle \theta, x \rangle = s} f(x) dx = Rf(\theta, s)$. ■

Now we can prove the result about linear transformations.

Proposition 2.48 (Linear transformation theorem). *Let T be a nonsingular linear transformation of $\mathbb{R}^n$ with transpose denoted by T^* . Then*

$$R(f \circ T)(\theta, s) = \frac{1}{|\det T|} \frac{1}{|(T^*)^{-1}(\theta)|} Rf\left(\frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}, \frac{s}{|(T^*)^{-1}(\theta)|}\right)$$

Remark 2.49. If T is an orthogonal transformation and θ is a unit vector, then $(T^*)^{-1}(\theta)$ is also a unit vector. However, this is not true for a general linear transformation, and this accounts for the more complicated form here versus the form in the rotation theorem.

Proof. Recall that the defining property of the transpose is that for all $x, y \in \mathbb{R}^n$ we have

$$\langle Tx, y \rangle = \langle x, T^*y \rangle.$$

Let D_s be the set of all points x in $\mathbb{R}^n$ which satisfy the inequality $\langle \theta, x \rangle \leq s$. Also let E_s be the set of all points y in $\mathbb{R}^n$ which satisfy the inequality $\langle (T^*)^{-1}(\theta), y \rangle \leq s$. Then we can prove that the image of D_s under T satisfies the equation

$$T(D_s) = E_s. \quad (2.15)$$

In fact equation (2.15) is equivalent to $D_s = T^{-1}(E_s)$, because T is nonsingular. But $x \in D_s$ if and only if $\langle \theta, x \rangle \leq s$, which of course happens if and only if $\langle \theta, T^{-1}Tx \rangle = \langle (T^{-1})^*\theta, Tx \rangle \leq s$. By the definition of E_s , this last inequality holds if and only if $Tx \in E_s$. This is equivalent to $x \in T^{-1}(E_s)$ and this establishes equation (2.15).

Now apply the change of variables theorem with the substitution defined by T . Note that the Jacobian of T is just $|\det T|$, so we get

$$\begin{aligned} \int_{T(D_s)} f(y) dy &= \int_{D_s} f(T(x)) JT(x) dx \\ &= |\det T| \int_{D_s} (f \circ T)(x) dx. \end{aligned} \quad (2.16)$$

Note that $(T^*)^{-1}(\theta)$ is not necessarily a unit vector. This is not a problem, however, because the set $T(D_s)$ is the same as the set E_s and it is easy to see that E_s is identical with the set of points $y \in \mathbb{R}^n$ such that

$$\left\langle \frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}, y \right\rangle \leq \frac{s}{|(T^*)^{-1}(\theta)|}.$$

Here we see that $\frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}$ is a unit vector. Therefore, equation (2.16) leads to

$$\int_{\left\langle \frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}, y \right\rangle \leq \frac{s}{|(T^*)^{-1}(\theta)|}} f(y) dy = |\det T| \int_{\langle \theta, x \rangle \leq s} (f \circ T)(x) dx.$$

This can be rephrased in terms of the operator M as

$$Mf \left(\frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}, \frac{s}{|(T^*)^{-1}(\theta)|} \right) = |\det T| M(f \circ T)(\theta, s).$$

If we now differentiate both sides of this equation with respect to s and use lemma 2.47 then we get

$$\frac{1}{|(T^*)^{-1}(\theta)|} Rf \left(\frac{(T^*)^{-1}(\theta)}{|(T^*)^{-1}(\theta)|}, \frac{s}{|(T^*)^{-1}(\theta)|} \right) = |\det T| R(f \circ T)(\theta, s)$$

which establishes the conclusion. ■

A dilation of a function $f(x)$ is a function of the form $f(\gamma x)$, where γ is a nonzero real number. Therefore we define an operator δ_γ by $\delta_\gamma f(x) = f(\gamma x)$ and we call δ_γ the **dilation operator** with factor γ .

Proposition 2.50 (Dilation theorem). *Let $\theta \in S^{n-1}$, let $s \in \mathbb{R}$, and let f be integrable on $\mathbb{R}^n$. Then if $\gamma \neq 0$ we have*

$$R(\delta_\gamma f)(\theta, s) = \gamma^{1-n} Rf(\theta, \gamma s)$$

Proof. (sketch). If $\gamma > 0$, this is an easy consequence of the previous theorem with $T(x) = \gamma x$. In this case the matrix of T is γI and hence $\det T = \gamma^n$, $T = T^*$, and $T^{-1} = \frac{1}{\gamma} T$. If $\gamma < 0$ we can use the fact that $\gamma = \text{sgn}(\gamma)|\gamma|$ to achieve the same result. ■

In Chapter 1 we gave the name “convolution” to the filtering process required for reconstructing a function from its projections. In this chapter we can precisely define the concept.

Definition 2.51 (Convolution). If f and g are integrable functions on $\mathbb{R}^n$, then we define the **convolution** $f * g$ of f and g by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy$$

If F and G are integrable functions on the cylinder Z^n then we define their convolution as

$$(F * G)(\theta, s) = \int_{t \in \mathbb{R}} F(\theta, s - t) g(t) dt$$

Note that the integration in this case is only with respect to the second variable.

The question arises as to which functions can be convolved successfully. By using Fubini’s theorem and the change of variables theorem we can show that the convolution of integrable functions is again integrable. To show this let f and g be integrable functions on $\mathbb{R}^n$ and define $h = f * g$ according to definition 2.51. We now use Fubini’s theorem on $\mathbb{R}^n \times \mathbb{R}^n$ as follows: in the current context x' and x'' will both denote elements of $\mathbb{R}^n$ so (x', x'') will be a typical element of $\mathbb{R}^n \times \mathbb{R}^n$. We can then write definition 2.51 in the form

$$h(x') = \int_{x'' \in \mathbb{R}^n} f(x' - x'') g(x'') dx''$$

Let

$$K = \int_{\mathbb{R}^n} |f(u)| du$$

Because f is integrable, K is a finite real number. Then, note that in the following integration x'' is constant,

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x' - x'') g(x'')| dx' &= |g(x'')| \int_{\mathbb{R}^n} |f(x' - x'')| dx' \\ &= |g(x'')| \int_{\mathbb{R}^n} |f(x')| dx' = K |g(x'')|. \end{aligned}$$

Here, we used the invariance of integration under translations, which was proved in corollary 2.23, to drop the term x'' in the integral. We thus have

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x' - x'') g(x'')| dx' \right) dx'' &= \int_{\mathbb{R}^n} K |g(x'')| dx'' \\ &= K \int_{\mathbb{R}^n} |g(x'')| dx'' < \infty \end{aligned}$$

by the hypothesis that g is integrable. Thus the function $x'' \mapsto \int_{\mathbb{R}^n} |f(x' - x'') g(x'')| dx'$ is integrable for all $x' \in \mathbb{R}^n$. But by the absolute integrability of the Lebesgue theory this means that also the function $x'' \mapsto \int_{\mathbb{R}^n} f(x' - x'') g(x'') dx'$ is integrable, so by the second part of Fubini's theorem, the function $(x', x'') \mapsto f(x' - x'') g(x'')$ is integrable on $\mathbb{R}^{2n}$. Now, by the first part of Fubini's theorem, the function that takes x' to $\int_{\mathbb{R}^n} f(x' - x'') g(x'') dx''$ is integrable and this is precisely the function $h = f * g$. We thus have proved that the convolution is integrable.

Now that we know that convolutions of integrable functions are integrable, we can investigate the effect of convolution on the Radon transform.

Proposition 2.52 (Convolution theorem for the Radon transform). *If f and g are integrable on $\mathbb{R}^n$, then so is $f * g$ and*

$$R(f * g)(\theta, s) = Rf(\theta, s) * Rg(\theta, s)$$

Proof. Before beginning the proof, recall our convention that anytime there is a variable of the form $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ we let $u' = (u_1, \dots, u_{n-1}) \in \mathbb{R}^{n-1}$.

We start by taking an orthonormal basis $\{\theta_1, \theta_2, \dots, \theta_{n-1}, \theta\}$ of $\mathbb{R}^n$ containing θ . Then we use the definition of the Radon transform and the definition of convolution to get

$$\begin{aligned} R(f * g)(\theta, s) &= \int_{x' \in \mathbb{R}^{n-1}} (f * g)(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta) dx' \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} f(x_1\theta_1 + x_2\theta_2 + \dots + x_{n-1}\theta_{n-1} + s\theta - w) g(w) dw dx' \end{aligned}$$

Make the orthogonal change of variables

$$w = y_1\theta_1 + y_2\theta_2 + \dots + y_{n-1}\theta_{n-1} + t\theta$$

in which the differential dw becomes $dy'dt$. This is possible by corollary 2.24 because the integrand in the inner integral is Lebesgue integrable for almost all w . From this we get

$$\begin{aligned} R(f * g)(\theta, s) &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f((x_1 - y_1)\theta_1 + \cdots + (x_{n-1} - y_{n-1})\theta_{n-1} + (s - t)\theta) \\ &\quad \times g(y_1\theta_1 + y_2\theta_2 + \cdots + y_{n-1}\theta_{n-1} + t\theta) dy' dt dx' \end{aligned} \quad (2.17)$$

The integration in the innermost integral is translation invariant, so we can replace $x_j - y_j$ by x_j . Also, because the integrand in equation (2.17) is integrable, Fubini's theorem implies that we can interchange the order of integration. Taking this into account yields

$$\begin{aligned} R(f * g)(\theta, s) &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} f(x_1\theta_1 + \cdots + x_{n-1}\theta_{n-1} + (s - t)\theta) dx' \\ &\quad \times g(y_1\theta_1 + y_2\theta_2 + \cdots + y_{n-1}\theta_{n-1} + t\theta) dy' dt \end{aligned}$$

The innermost integral is

$$\int_{\mathbb{R}^{n-1}} f(x_1\theta_1 + \cdots + x_{n-1}\theta_{n-1} + (s - t)\theta) dx'$$

and we recognize this as $Rf(\theta, s - t)$, which is independent of y and hence factors through the integration with respect to the differential dy' . Hence, the previous equation becomes

$$\begin{aligned} &\int_{\mathbb{R}} \left[\int_{\mathbb{R}^{n-1}} g(y_1\theta_1 + y_2\theta_2 + \cdots + y_{n-1}\theta_{n-1} + t\theta) dy' \right] Rf(\theta, s - t) dt \\ &= \int_{\mathbb{R}} Rg(\theta, t) Rf(\theta, s - t) dt \\ &= Rf(\theta, s) * Rg(\theta, s) \end{aligned}$$

Here we used the fact that the integral in brackets is precisely $Rg(\theta, t)$. The proof is complete. ■

An important connection exists between the Radon transform and the Fourier transform.

Definition 2.53 (Fourier transform). If f is an integrable function on $\mathbb{R}^n$ then we define

$$(\mathcal{F}f)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, y \rangle} dx$$

The function $\mathcal{F}f$ is called the **Fourier transform** of f and is also denoted by $\widehat{f}$. If $expr$ denotes a long expression, then we may use the notation $(expr)^\wedge$ to denote the Fourier transform of $expr$.

Note that the Fourier transform can be defined on any Euclidean space. If it is necessary to emphasize the dimension, then we will write $\mathcal{F}_n$, but usually the context determines the correct dimension.

Remark 2.54. Some authors use $\widehat{f}$ to denote the Radon transform of f and the notation $\widetilde{f}$ to denote the Fourier transform. However, it is more common outside the field of tomography to use $\widehat{f}$ for the Fourier transform and we follow this convention.

Remark 2.55. The absolute value of the integrand in the definition of the Fourier transform is $|f(x)|$, since $|e^{-i\langle x, y \rangle}| = 1$. Because we know that f is integrable, this shows that the Fourier transform of an integrable function exists for all y in $\mathbb{R}^n$.

For our purposes we need just one property of the Fourier transform: the **uniqueness property of the Fourier transform**: If $\mathcal{F}f = 0$, then $f = 0$ almost everywhere.

The Fourier transform has many other interesting and useful properties. Readers who have not encountered the Fourier transform can find a proof of the uniqueness property in the first few pages of Stein and Weiss [583]. This reference has many more applications of the Fourier transform as do Dym and McKean [138] and Hewitt and Stromberg [304], among numerous excellent references.

The relation between the Radon transform and the Fourier transform is sometimes, called the Fourier slice theorem and sometimes, the projection theorem. We call it the **slice-projection theorem**.

Theorem 2.56 (Slice projection). *If f is integrable on $\mathbb{R}^n$ and if θ is a unit vector, then*

$$\mathcal{F}_1 R_\theta f(s) = (2\pi)^{\frac{n-1}{2}} \mathcal{F}_n f(s\theta) \quad (2.18)$$

Proof. By the hypothesis that f is integrable on $\mathbb{R}^n$ we see that the n -dimensional Fourier transform on the right of equation (2.18) is defined and that by the existence of projections theorem, proposition 2.38, the one-dimensional Fourier transform on the left of the equation is defined.

If we now look at the definition of the Fourier transform, the integration over $\mathbb{R}^n$ can be carried out by integrating over any set of parallel hyperplanes. In particular, this could be done over the set of hyperplanes orthogonal to θ . This is a consequence of Fubini's theorem and the result is

$$\mathcal{F}_n f(s\theta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}} \left[\int_{\langle \theta, x \rangle = p} f(x) e^{-i\langle x, s\theta \rangle} dx \right] dp$$

But if x is in the hyperplane $\langle \theta, x \rangle = p$, then $\langle x, s\theta \rangle = s \langle x, \theta \rangle = sp$. Hence, the exponential term is constant in the inner integral, so

$$\mathcal{F}_n f(s\theta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}} \left[\int_{\langle \theta, x \rangle = p} f(x) dx \right] e^{-isp} dp$$

We recognize the inner integral to be the Radon transform, hence,

$$\begin{aligned} \mathcal{F}_n f(s\theta) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}} R_\theta f(p) e^{-isp} dp \\ &= (2\pi)^{-\frac{n-1}{2}} \mathcal{F}_1 R_\theta f(s) \end{aligned}$$

The constant $(2\pi)^{-\frac{n-1}{2}}$ arises because the one-dimensional Fourier transform needs to borrow the factor $(2\pi)^{-\frac{1}{2}}$ from $(2\pi)^{-\frac{n}{2}}$. ■

The name of the slice projection theorem comes from the fact, just established, that restricting the Fourier transform of f to the slice of $\mathbb{R}^n$ along the direction θ yields the Fourier transform of the Radon projection orthogonal to θ .

Using the properties of the Fourier transform stated above, we can give the following important uniqueness result.

Theorem 2.57 (Uniqueness theorem for the Radon transform). *An integrable function is uniquely determined by its Radon transform: if f and g are integrable functions on $\mathbb{R}^n$ with*

$$Rf(\theta, s) = Rg(\theta, s) \quad (2.19)$$

for all unit vectors θ and all $s \in \mathbb{R}$, then

$$f(x) = g(x)$$

for almost all $x \in \mathbb{R}^n$.

Proof. Let $h(x) = f(x) - g(x)$. By the linearity of the Radon transform and the hypothesis (2.19) we see that

$$R_\theta h = 0$$

for all unit vectors θ , from which it is immediately known that the Fourier transform of $R_\theta h$ is also zero for all θ . By the slice-projection theorem we see that for all $\theta \in S^{n-1}$ and all $s \in \mathbb{R}$ we have

$$\mathcal{F}_n h(s\theta) = 0$$

But any $x \in \mathbb{R}^n$ is of the form $s\theta$ (e.g., if $x \neq 0$, then we can take $s = |x|$ and $\theta = \frac{x}{|x|}$), so we end up with $\mathcal{F}_n h(x) = 0$ for all $x \in \mathbb{R}^n$. From this and the uniqueness property of the Fourier transform it follows that $h(x) = 0$ for almost all $x \in \mathbb{R}^n$. The definition of h then implies the conclusion of the theorem. ■

Note that the uniqueness theorem does not give an inversion formula for the Radon transform. An inversion formula would accept the Radon transform as input data and would yield the original function as output. Those readers who are familiar with the inverse Fourier transform will naturally wonder whether this can be applied to give an explicit inversion formula for the Radon transform. This can be done but, unfortunately, not directly. One problem is that not every integrable function has an integrable Fourier transform. Even such a simple integrable function as the characteristic function of the unit square has a nonintegrable Fourier transform.

We will reproduce Radon's original inversion formula from his 1917 paper in section 2.6. Some readers may want to consult the excellent introductory paper of Strichartz [585] in which a formal inversion⁴ of the Radon transform via the Fourier transform is presented. In a like manner Nievergelt [449, 450] gives an elementary inversion formula. Nievergelt's formula is closely related to the filtered backprojection method that is discussed in section 2.7.

Remark 2.58. There are many other inversion formulas and uniqueness results for Radon transforms. These are presented in chapter 3.

2.4 A Homogeneous Extension of the Radon Transform

It is useful to extend the domain of Rf from the cylinder $Z = S^{n-1} \times \mathbb{R}$ to $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$. This yields a different parameterization of hyperplane integrals. If the extension is done in such a way as to make the Radon transform homogeneous of degree -1 , then this extension corresponds to the Radon transform developed by Gel'fand, Graev, and Vilenkin [194], compare chapter 3 of Gel'fand and Shilov [195].

Definition 2.59. A function $F(\xi, s)$ defined on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ is said to be *homogeneous* of degree k if $F(a\xi, as) = a^k F(\xi, s)$ for $a > 0$.

Given a function $h(\theta, s)$ defined on $S^{n-1} \times \mathbb{R}$, we can extend h to $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ by defining $H(\xi, s) = |\xi|^k h(\frac{\xi}{|\xi|}, \frac{s}{|\xi|})$ for $\xi \in \mathbb{R}^n \setminus \{0\}, s \in \mathbb{R}$. It is easy to check that H is an extension of h which is homogeneous of degree k .

We can extend the Radon transform in this way:

Definition 2.60. If $\xi \in \mathbb{R}^n \setminus \{0\}$, then we define $Gf(\xi, s) = \frac{1}{|\xi|} Rf(\frac{\xi}{|\xi|}, \frac{s}{|\xi|})$.

The proof of the next result follows directly from the definitions and the evenness of the Radon transform.

Proposition 2.61. *The extended Radon transform G has the following symmetry property: Gf is an even function.*

$$Gf(-\xi, -s) = Gf(\xi, s)$$

⁴ A "formal" inversion derives the inversion formula without regard to the validity of integrations and other limit operations.

Also G is homogeneous of degree -1 : if a is a nonzero real number then

$$Gf(a\xi, as) = |a|^{-1} Gf(\xi, s)$$

Finally, G agrees with R on the unit sphere: if $\theta \in S^{n-1}$ then

$$Gf(\theta, s) = Rf(\theta, s)$$

The proof of the next result is omitted because it involves techniques beyond the prerequisites for this chapter.

Theorem 2.62. *If $(\xi, s) \in (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$, and $\xi_n \neq 0$, then*

$$Gf(\xi, s) = \frac{1}{|\xi_n|} \int_{\mathbb{R}^{n-1}} f\left(x_1, \dots, x_{n-1}, \frac{1}{\xi_n} (s - (\xi_1 x_1 + \dots + \xi_{n-1} x_{n-1}))\right) dx'.$$

It can be shown that the integral defining $Gf(\xi, s)$ gives the hyperplane integral of f over the hyperplane defined by the equation $\langle x, \xi \rangle = s$. In the particular case that ξ is a unit vector θ , we obtain the following.

Corollary 2.63. *If $\theta \in S^{n-1}$, then $Gf(\theta, s) = Rf(\theta, s)$.*

Therefore, Gf is a homogeneous extension of degree -1 of the Radon transform from the cylinder to $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$. The operator G is the version of the Radon transform studied by Gel'fand and coworkers [194, 195].

Remark 2.64. One interpretation of this theorem is that we have a new parametrization of the Radon transform. Let ξ be a unit vector. Formerly we computed the hyperplane integral over $\langle \xi, x \rangle = s$ by parametrizing the hyperplane by

$$x_1 \alpha_1 + x_2 \alpha_2 + \dots + x_{n-1} \alpha_{n-1} + s \xi$$

where the α_j values are chosen so that $\{\alpha_1, \alpha_2, \dots, \alpha_{n-1}, \xi\}$ is an orthonormal basis of $\mathbb{R}^n$. The Gel'fand, Graev, and Vilenkin formulation yields the alternate parametrization

$$\left(x_1, \dots, x_{n-1}, \frac{1}{\xi_n} (s - (\xi_1 x_1 + \dots + \xi_{n-1} x_{n-1})) \right).$$

2.5 Examples of the Radon Transform

2.5.1 The Radon Transform on $\mathbb{R}^2$

Let f be an integrable function on $\mathbb{R}^2$. For any hyperplane (i.e., line) of the form $\langle x, \theta \rangle = s$ in $\mathbb{R}^2$, we can find a real number α between 0 and 2π such that $\theta = (\cos \alpha, \sin \alpha)$. Define $\theta^\perp = (-\sin \alpha, \cos \alpha)$. Note that $\{\theta, \theta^\perp\}$ form an orthonormal basis of $\mathbb{R}^2$ and that $\{\theta^\perp\}$ forms a basis of the linear space (line through the origin in this case) which is parallel to the hyperplane $\langle x, \theta \rangle = s$. Then any point on this hyperplane can be expressed as

$$t\theta^\perp + s\theta = (s \cos \alpha - t \sin \alpha, s \sin \alpha + t \cos \alpha)$$

where t is a real number. According to the definition of integration over hyperplanes, (2.26) with $\theta_1 = \theta^\perp$, we get

$$\begin{aligned} Rf(\theta, s) &= \int_{\langle x, \theta \rangle = s} f(x) dx \\ &= \int_{\mathbb{R}} f(t\theta^\perp + s\theta) dt \\ &= \int_{-\infty}^{\infty} f(s \cos \alpha - t \sin \alpha, s \sin \alpha + t \cos \alpha) dt. \end{aligned}$$

This last expression is useful in practical calculations of the Radon transform on $\mathbb{R}^2$.

2.5.2 The Radon Transform on $\mathbb{R}^3$

Let f be an integrable function on $\mathbb{R}^3$. Any unit vector $\theta \in \mathbb{R}^3$ can be expressed in spherical coordinates as $\theta = (\sin \phi \cos \beta, \sin \phi \sin \beta, \cos \phi)$ for some real numbers ϕ and β . We define

$$\begin{aligned} \theta_1 &= (\cos \phi \cos \beta, \cos \phi \sin \beta, -\sin \phi) \\ \theta_2 &= (-\sin \beta, \cos \beta, 0) \end{aligned}$$

Then it is easy to check that $\{\theta_1, \theta_2, \theta\}$ is an orthonormal basis of $\mathbb{R}^3$ and that $\{\theta_1, \theta_2\}$ is an orthonormal basis of the linear space $\theta^\perp$ orthogonal to θ . Therefore, if f is an integrable function on $\mathbb{R}^3$, then f restricted to the hyperplane $\langle x, \theta \rangle = s$ has the form

$$f(r\theta_1 + t\theta_2 + s\theta),$$

where r and t are real numbers. We can then expand this when we integrate to get the Radon transform:

$$\begin{aligned} Rf(\theta, s) &= \int_{\langle x, \theta \rangle = s} f(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r\theta_1 + t\theta_2 + s\theta) dr dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r \cos \phi \cos \beta - t \sin \beta + s \sin \phi \cos \beta, r \cos \phi \sin \beta \\ &\quad + t \cos \beta + s \sin \phi \sin \beta, -r \sin \phi + s \cos \phi) dr dt \end{aligned}$$

2.5.3 The Radon Transform of Some Common Functions

In this section we derive formulas for the Radon transform of Gaussian functions, balls and squares. When we talk about the Radon transform of a set we really mean the Radon transform of the function which is identically 1 on the set with other values equal to zero. Such a function is called the **characteristic function** of the set.

Proposition 2.65 (Gaussian functions). Let $a > 0$ and let $\gamma_a(x) = \exp(-a|x|^2)$ for $x \in \mathbb{R}^n$. Then

$$R \gamma_a(\theta, s) = \left(\frac{\pi}{a}\right)^{\frac{n-1}{2}} \exp(-as^2)$$

Proof. It is clear from the definition that γ_a is a radial function. By the radial function theorem, corollary 2.43, it is enough to show that $R \gamma_a(e_n, s) = \left(\frac{\pi}{a}\right)^{\frac{n-1}{2}} \exp(-as^2)$. Here e_n is the n th standard unit vector and the hyperplane is $x_n = 0$, so

$$\begin{aligned} R \gamma_a(\theta, s) &= \int_{x' \in \mathbb{R}^{n-1}} \exp(-a|x'|^2 - as^2) dx' \\ &= \exp(-as^2) \int_{-\infty}^{\infty} \exp(-ax_1^2) dx_1 \cdots \int_{-\infty}^{\infty} \exp(-ax_{n-1}^2) dx_{n-1} \end{aligned}$$

We used the laws of exponents and also Fubini's theorem (to iterate the integration). These integrals over $(-\infty, \infty)$ are all the same and there are $n - 1$ factors. The result now follows from the classical result that $\int_{-\infty}^{\infty} \exp(-ax_1^2) dx_1 = \sqrt{\frac{\pi}{a}}$. ■

Proposition 2.66. Let β_r be the characteristic function of the ball of radius r centered at the origin of $\mathbb{R}^n$. Then

$$R \beta_r(\theta, p) = \begin{cases} \frac{(\pi(r^2 - p^2))^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} & \text{if } |p| \leq r \\ 0 & \text{if } |p| > r \end{cases}$$

In particular, in dimension $n = 2$ we have

$$R \beta_r(\theta, p) = \begin{cases} 2\sqrt{r^2 - p^2} & \text{if } |p| \leq r \\ 0 & \text{if } |p| > r \end{cases}$$

and in dimension $n = 3$

$$R \beta_r(\theta, p) = \begin{cases} \pi(r^2 - p^2) & \text{if } |p| \leq r \\ 0 & \text{if } |p| > r \end{cases}$$

Proof. The ball of radius r centered at the origin can be defined as the set of points in $\mathbb{R}^n$ satisfying the equation

$$|x| < r \tag{2.20}$$

It is easy to see from the symmetry of the ball that its characteristic function is radial, so we can use the radial function theorem, corollary 2.43. This means that we can find the Radon transform for all directions by looking only at hyperplanes orthogonal to a standard unit vector. Such a hyperplane has the form $x_n = p$, so equation (2.20)

becomes

$$\sqrt{x_1^2 + \cdots x_{n-1}^2 + p^2} \leq r \text{ which is equivalent to}$$

$$\sqrt{x_1^2 + \cdots x_{n-1}^2} \leq \sqrt{r^2 - p^2}$$

This is valid, of course, only for $|p| \leq r$. For other values of p an empty intersection exists between the ball and the hyperplane, so the Radon transform will be zero for $|p| > r$. The last equation is that of a ball of radius $\sqrt{r^2 - p^2}$ in $\mathbb{R}^{n-1}$. The hyperplane integral of the characteristic function then becomes the volume of this $(n-1)$ ball which is

$$\left(\sqrt{r^2 - p^2}\right)^{n-1} \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$$

(compare theorem 2.16 for the volume of a ball). Hence, we see that $R\beta_r(\theta, p) = \frac{(\pi(r^2 - p^2))^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}$ if $|p| \leq r$ with the value 0 for other p . ■

In the next result we use a slight abuse of language in which the unit vector θ and its angle share the same symbol.

Proposition 2.67. *Let σ_a be the characteristic function of the square of side $2a$ centered at the origin ($a > 0$). Then,*

$$\text{if } \theta \in \left(0, \frac{\pi}{4}\right],$$

$$R\sigma_a(\theta, p) = \begin{cases} 2a \sec \theta & \text{if } |p| \leq a(\cos \theta - \sin \theta) \\ \frac{a \cos \theta + a \sin \theta - |p|}{\sin \theta \cos \theta} & \text{if } a(\cos \theta - \sin \theta) < |p| \leq a(\cos \theta + \sin \theta) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{if } \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right),$$

$$R\sigma_a(\theta, p) = \begin{cases} 2a \csc \theta & \text{if } |p| \leq a(\sin \theta - \cos \theta) \\ \frac{a \cos \theta + a \sin \theta - |p|}{\sin \theta \cos \theta} & \text{if } a(\sin \theta - \cos \theta) < |p| \leq a(\cos \theta + \sin \theta) \\ 0 & \text{elsewhere} \end{cases}$$

$$\text{if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right],$$

$$R\sigma_a(\theta, p) = \begin{cases} 2a \csc \theta & \text{if } |p| \leq a(\sin \theta + \cos \theta) \\ \frac{|p| - a \sin \theta + a \cos \theta}{\sin \theta \cos \theta} & \text{if } a(\sin \theta + \cos \theta) < |p| \leq a(\sin \theta - \cos \theta) \\ 0 & \text{elsewhere} \end{cases}$$

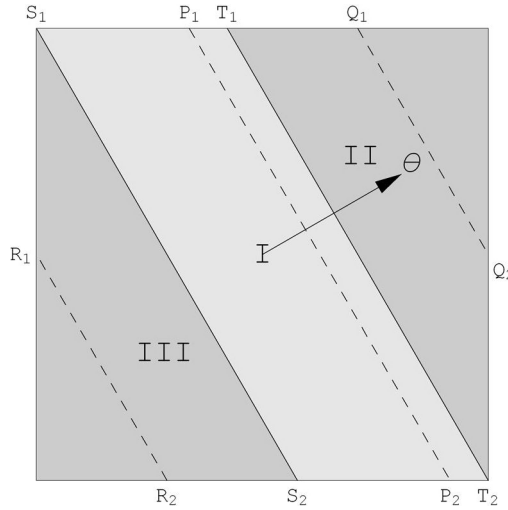
if $\theta \in \left(\frac{3\pi}{4}, \pi\right)$,

$$R\sigma_a(\theta, p) = \begin{cases} 2a |\sec \theta| & \text{if } |p| \leq a(\sin \theta + \cos \theta) \\ \frac{|p| - a \sin \theta + a \cos \theta}{\sin \theta \cos \theta} & \text{if } a(\sin \theta + \cos \theta) < |p| \leq a(\sin \theta - \cos \theta) \\ 0 & \text{elsewhere} \end{cases}$$

Remark 2.68. The translation theorem for the Radon transform can be used with this result to obtain the Radon transform for squares not centered at the origin. The linear transformation theorem can also be applied to the current result to obtain the Radon transform of rectangles and parallelograms.

Remark 2.69. If $\theta = 0, \frac{\pi}{2}$, or π , then it is easy to check that $R\sigma_a(\theta, p) = 2a$ if $|p| \leq a$ and is zero elsewhere.

Sketch of the proof. First let θ be in the range from 0 to $\frac{\pi}{4}$. By definition of characteristic function, σ_a has the value 1 on the square $[-a, a] \times [-a, a]$ and the value 0 elsewhere. Therefore, every line integral will be the length of the intersection of the line with the square and these integrals will determine the Radon transform. The following diagram illustrates a typical case.



In the diagram we see three typical regions of the square: region I, region II, and region III. We also see the unit vector θ issuing from the origin and five lines orthogonal to θ . Two of these lines, S_1S_2 and T_1T_2 are boundaries of region I. All the lines satisfy the equation

$$x \cos \theta + y \sin \theta = p \quad (2.21)$$

where p is the signed distance of the line from the origin.

First, every line orthogonal to θ in region I has a value of p satisfying

$$|p| < a \cos \theta - a \sin \theta. \quad (2.22)$$

This is because the extremes for p in this region are attained at the lines $T_1 T_2$ and $S_1 S_2$. But it is evident from the diagram that $T_2 = (a, -a)$ so equation (2.21) implies that

$$p = a \cos \theta - a \sin \theta$$

for points on the line $T_1 T_2$. A similar calculation with $S_1 = (-a, a)$ shows that

$$p = -a \cos \theta + a \sin \theta$$

for points on the line $S_1 S_2$. This is exactly the negative of the value for the line $T_1 T_2$, so we then have the desired bounds in (2.22).

In a similar fashion we can determine that in region II all the lines orthogonal to θ satisfy

$$a \cos \theta - a \sin \theta \leq p < a \cos \theta + a \sin \theta \quad (2.23)$$

and that in region III

$$-a \cos \theta - a \sin \theta < p \leq -a \cos \theta + a \sin \theta. \quad (2.24)$$

Inequalities (2.23) and (2.24) clearly coalesce to the single inequality

$$a \cos \theta - a \sin \theta \leq |p| < a \cos \theta + a \sin \theta \quad (2.25)$$

Now take a typical line segment $P_1 P_2$ orthogonal to θ in region I. Let $P_1 = (x_1, y_1)$. From the diagram we see that $y_1 = a$ and then from equation (2.21) we get $x_1 = p \sec(\theta) - \tan(\theta)$. The same analysis applies to P_2 and in the end we arrive at

$$\begin{aligned} P_1 &= (p \sec(\theta) - a \tan(\theta), a) \\ P_2 &= (p \sec(\theta) + a \tan(\theta), -a). \end{aligned}$$

An easy calculation then gives

$$R\sigma_a(\theta, p) = |P_1 P_2| = 2a \sec \theta.$$

The same reasoning applies in regions II and III; we can find expressions for the endpoints of segments $Q_1 Q_2$ and $R_1 R_2$, compute the lengths, and thus obtain the value of the Radon transform.

Piecing all this together gives the desired result for θ in the range 0 to $\frac{\pi}{4}$.

Next let θ be in the range from $\frac{\pi}{4}$ to $\frac{\pi}{2}$. As the reader can easily show by manipulating the preceding diagram, a rotation through negative 90° followed by a reflection about the x axis will transform the calculation of the Radon transform into the range 0 to $\frac{\pi}{4}$. Using the linear transformation theorem, we observe that the effect of this transformation is to interchange $\sin$ with $\cos$ (and, of course, as a consequence, $\sec$ with $\csc$). Making

these replacements in the formula for $R\sigma_\alpha$ in the range 0 to $\frac{\pi}{4}$ yields the formula stated for the range $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

To obtain the Radon transform in the range $\frac{\pi}{2}$ to π we again take a linear transformation – this time a rotation through negative 90° . The reader can check that the stated formulas follow from the case of the range 0 to $\frac{\pi}{2}$.

By the symmetry theorem, Proposition 2.39, there is no need to do separate calculations for θ in the range π to 2π because, if δ is any direction with an angle between π and 2π , $-\delta$ has an angle between 0 and π . ■

2.6 Inversion, Reconstruction, and Approximate Identities

In this and the next few sections we investigate the possibility of recovering a function from its Radon transform. This was treated informally in chapter 1 where the Radon transform was obtained via x-rays through the function. We saw that an approximate inversion formula was possible by using a filtered version of backprojection. In section 2.7 we will precisely define the backprojection operator and derive a theorem justifying the filtered backprojection process. Any such process that yields an exact or approximate version of the unknown function based on the knowledge of its Radon transform is called “reconstruction of a function from its Radon transform” or, in brief, *reconstruction*.

Inversion refers to the possibility of finding a linear operator S with the property that $S(Rf) = f$. If we can do this inversion, then we get an exact reconstruction of f because knowing the formula for f and the values of Rf clearly lead to a formula for f . The general problem of inversion will be treated in chapter 3, although in Section 2.8 of this chapter we present the inversion method developed by Radon in his 1917 paper [508]. In this chapter we also present a rigorous development of the filtered backprojection method for the approximate inversion of the Radon transform.

But first we need a result on approximate identities. This result was used without proof or reference by Radon, so we present a detailed discussion and derivation.

2.6.1 Approximate Identities

The lemma on approximate identities shows how a bounded, continuous, real valued function can be approximated by certain convolutions. It is a useful tool in both inversion and reconstruction.

The linear space $L^1(\mathbb{R})$ of Lebesgue integrable real valued functions can be considered as a commutative algebra if multiplication is defined as convolution. However, this algebra has no multiplicative identity; there is no element ϕ with the property that $f * \phi = f$ for all f in the algebra. We can get around this inconvenience by introducing a family of functions ϕ_ε , defined for $\varepsilon > 0$, such that $f * \phi_\varepsilon \approx f$ for all f in the algebra. The meaning of the approximation will be made precise in the next lemma. Such a family of functions is called an *approximate identity*. However, do such

approximate identities exist? The next lemma shows that they do. Subsequent results show their utility, which may not be apparent just now.

Lemma 2.70 (Approximate identity theorem). *Let φ be a nonnegative, integrable function defined on $(-\infty, \infty)$ such that*

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1 \quad (2.26)$$

For each $\varepsilon > 0$ define

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right) \quad (2.27)$$

Then for any continuous, bounded function f , we have

$$\lim_{\varepsilon \rightarrow 0} f * \varphi_\varepsilon(x) = f(x)$$

for all real numbers x .

Proof. Because f is assumed to be bounded, we can choose an upper bound M for $|f|$. Because the integral in equation (2.26) is finite, for any real number $\delta > 0$ we can find a real number $a > 0$ such that

$$\int_{|x| > a} \varphi(x) dx < \frac{\delta}{4M}. \quad (2.28)$$

This is a consequence of the fact that the tails of an absolutely convergent integral must tend to zero. Then using the substitution $x = \frac{y}{\varepsilon}$ we get

$$\int_{|y| > \varepsilon a} \varphi_\varepsilon(y) dy < \frac{\delta}{4M}. \quad (2.29)$$

The same substitution shows that

$$\int_{-\infty}^{\infty} \varphi_\varepsilon(y) dy = 1, \quad (2.30)$$

so, letting x represent a fixed real number, we have

$$f(x) = \int_{-\infty}^{\infty} f(x) \varphi_\varepsilon(y) dy \quad (2.31)$$

(Note that $f(x)$ is constant with respect to the integration; equation (2.31) then follows immediately from equation (2.30)). Use this equation together with the definition of

the convolution to get

$$\begin{aligned} f * \varphi_\varepsilon(x) - f(x) &= \int_{-\infty}^{\infty} f(x-y) \varphi_\varepsilon(y) dy - \int_{-\infty}^{\infty} f(x) \varphi_\varepsilon(y) dy \\ &= \int_{-\infty}^{\infty} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy \end{aligned}$$

Break up this last integral into two pieces obtaining

$$\begin{aligned} f * \varphi_\varepsilon(x) - f(x) &= \int_{|y| > \varepsilon a} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy \\ &\quad + \int_{-\varepsilon a}^{\varepsilon a} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy \end{aligned} \quad (2.32)$$

For the first piece we can use the boundedness of f and equation (2.29) to get

$$\begin{aligned} \left| \int_{|y| > \varepsilon a} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy \right| &\leq 2M \int_{|y| > \varepsilon a} \varphi_\varepsilon(y) dy \\ &< \frac{\delta}{2} \end{aligned} \quad (2.33)$$

For the second piece we can use the second mean value theorem for integrals⁵ to get the existence of a real number $c = c(\varepsilon, a)$ in the interval $[-\varepsilon a, \varepsilon a]$ such that

$$\int_{-\varepsilon a}^{\varepsilon a} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy = (f(x-c) - f(x)) \int_{-\varepsilon a}^{\varepsilon a} \varphi_\varepsilon(y) dy \quad (2.34)$$

From this and equation (2.30) it follows immediately that

$$\left| \int_{-\varepsilon a}^{\varepsilon a} (f(x-y) - f(x)) \varphi_\varepsilon(y) dy \right| \leq |f(x-c) - f(x)| \quad (2.35)$$

because the nonnegativity of φ and equation (2.30) shows that

$$\left| \int_{-\varepsilon a}^{\varepsilon a} \varphi_\varepsilon(y) dy \right| \leq 1$$

⁵ The statement and proof of the second mean value theorem for integrals can be found in many elementary calculus books (e.g., Salas, Hille, and Etgen [540]).

Going back to equation (2.32) and using equations (2.33) and (2.35), we get

$$|f * \varphi_\varepsilon(x) - f(x)| \leq \frac{\delta}{2} + |f(x - c) - f(x)| \quad (2.36)$$

Because f is continuous at x , there exists some $\eta > 0$, η depending on δ , such that $|c| < \eta$ implies $|f(x - c) - f(x)| < \frac{\delta}{2}$. If we now choose any $\varepsilon < \frac{\eta}{a}$ we then have, because $c = c(\varepsilon, a)$ is in the interval $[-\varepsilon a, \varepsilon a]$, that $|c| < \eta$ so indeed we get $|f(x - c) - f(x)| < \frac{\delta}{2}$. Using this back in equation (2.36) shows that for any $\delta > 0$ there exists an $\eta > 0$ such that whenever $\varepsilon < \frac{\eta}{a}$ we have $|f * \varphi_\varepsilon(x) - f(x)| < \delta$. This proves that $\lim_{\varepsilon \rightarrow 0} f * \varphi_\varepsilon(x) = f(x)$. ■

Remark 2.71. The approximate identity theorem is valid on $\mathbb{R}^n$ with

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{x}{\varepsilon}\right). \quad (2.37)$$

Remark 2.72. The second mean value theorem for integrals is usually stated to require the hypothesis that both functions in the integrand are continuous. However, the proof easily extends to the case that φ is a nonnegative integrable function.

Remark 2.73. For those readers who are familiar with the L^p spaces we mention that a much more general result is possible. We can change the hypotheses so that f is now assumed to be an L^p function on $\mathbb{R}^n$. Then, if the convergence of $f * \varphi_\varepsilon$ is taken with respect to the L^p norm, we get the same result as the lemma. See Stein and Weiss [583], theorem 1.18, for details.

2.7 Backprojection, Filtered Backprojection, and Reconstruction

We now investigate approximate methods for reconstructing a function from its Radon transform. These are also called “approximate inversion methods.”

In chapter 1 we gave an intuitive introduction to the method of *filtered backprojection* for reconstructing a function from its Radon transform. This is also called the *convolution reconstruction method*. We now formalize the definition of backprojection, which is an important constituent of the filtered backprojection method. Before defining backprojection, let us review the procedure in chapter 1.

In chapter 1 we defined the backprojection $R_\theta^\# g$ of a function $g(\theta, p)$ defined on the cylinder Z^n . The function $R_\theta^\# g$ was obtained by smearing the values of $g(\theta, p)$ backward in the direction orthogonal to θ . In this smearing process all points x on the hyperplane $\langle x, \theta \rangle = p$ take on the value $g(\theta, p)$. In other words $R_\theta^\# g(x) = g(\theta, \langle x, \theta \rangle)$. This is a precise formula of the function $R_\theta^\# g$ that was described informally in chapter 1. In chapter 1 we then averaged these backprojections over all possible directions.

In this chapter we are more interested in the averaged backprojections than in backprojections in specific directions. Therefore, we do not bother to define the functions $R_\theta^\# g$; we go directly to the definition of the averaged backprojections. The resulting function is denoted by $R^\# g$ and is a function defined on $\mathbb{R}^n$. We know from elementary

calculus that the average of a function over an interval is proportional to the integral of the function. In our situation we do not care about the proportionality constant and we merely define the backprojection as the integral of the function $g(\theta, \langle x, \theta \rangle)$ over the unit sphere as follows:

Definition 2.74 (Backprojection). If $g : S^{n-1} \times \mathbb{R} \rightarrow \mathbb{C}$ we define the **backprojection** of g to be the function $R^\# g : \mathbb{R}^n \rightarrow \mathbb{C}$ defined by

$$R^\# g(x) = \int_{S^{n-1}} g(\theta, \langle x, \theta \rangle) d\theta$$

whenever the integral is defined. The backprojection operator is also called the **formal adjoint** of the Radon transform.

We need to use the concept of **measurable function** in a few places from now on. We will not give details on the definition of measurable functions; these can be found in the references for the Lebesgue theory: Hewitt and Stromberg [303], Strichartz [587], Evans and Garipey [153], Kuttler [375], and Stroock [588]. However, we will say that measurable functions are in the same relation to functions as measurable sets are to sets (compare the discussion on measurable sets, in section 2 of this chapter.) Just as there are many nonmeasurable sets, there are many nonmeasurable functions. On the other hand, just as nonmeasurable sets are hard to find in practice, the same is true of nonmeasurable functions. All the common functions encountered in tomography: integrable, continuous, pixel functions, etc., are measurable.

We will also need to use an extension of Fubini's theorem from $\mathbb{R}^k \times \mathbb{R}^p$ to $S^{n-1} \times \mathbb{R}^n$. This can be justified and a proof of a very general version of Fubini's theorem is given in the references cited for the Lebesgue theory. We will call this the **extended Fubini theorem**. The original Fubini theorem applied to integrable functions, so all the integrals came out to be finite. This is also true of the extended Fubini theorem, but the extended theorem also allows interchanges in the order of integration for any nonnegative measurable functions. The only thing is that in this case some or all of the integrals may be infinite.

The backprojection operator is called the formal adjoint of the Radon transform because it is the dual of the Radon transform in a sense analogous to the way the adjoint of a linear transformation on Euclidean space is dual to the original transformation.⁶ Although this duality does not play a role in this chapter, in subsequent chapters it plays a fruitful and even indispensable role. For now we just explain why backprojection is called the formal adjoint of the Radon transform.

In linear algebra the adjoint of a linear transformation $S : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is defined to be the unique linear transformation $S^* : \mathbb{R}^n \rightarrow \mathbb{R}^m$ which satisfies the equation

$$\langle Sx, y \rangle = \langle x, S^*y \rangle$$

⁶ Compare, Theorem 3.104 in chapter 3.

for all $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. This equation expresses the duality between the operators S and S^* . We wish to explore the possibility of a similar duality between the Radon transform and the backprojection operator.

Because the inner product in $\mathbb{R}^n$ is defined as a certain sum and because integrals are limits of sums, it is natural to define an inner product on $L^1(\mathbb{R}^n)$ by the formula

$$\langle f, g \rangle = \int_{\mathbb{R}^n} f(x) g(x) dx$$

We can define a similar inner product on the cylinder: if $u = u(\theta, p)$ and $w = w(\theta, p)$ then we define

$$\langle u, w \rangle = \int_{\mathbb{R}} \int_{S^{n-1}} u(\theta, p) w(\theta, p) d\theta dp$$

In analogy with the case S and S^* we would like to show that

$$\langle Rf, w \rangle = \langle f, R^\# w \rangle$$

Unfortunately, there are certain combinations of functions f and w for which the integrals defining the inner product do not have finite values. This is the reason that $R^\#$, the backprojection operator, is called the formal adjoint and not the adjoint of the Radon transform. However, we do have theorem 2.75.

Theorem 2.75 (Formal Adjoint Theorem). $\langle Rf, w \rangle = \langle f, R^\# w \rangle$.

Proof. Because we do not need this result in this chapter we proceed rather formally. We do the proof for nonnegative measurable functions, not caring much if the inner products sometimes come out to be infinite. In chapter 3 we revisit this result and derive a more general and precise version: theorem 3.29.

Because we are dealing with nonnegative measurable functions, the extended Fubini theorem provides the justification for the various changes in the order of integration.

$$\begin{aligned} \langle f, R^\# w \rangle &= \int_{\mathbb{R}^n} f(x) R^\# w(x) dx \\ &= \int_{\mathbb{R}^n} f(x) \int_{S^{n-1}} w(\theta, \langle x, \theta \rangle) d\theta dx \\ &= \int_{S^{n-1}} \int_{\mathbb{R}^n} f(x) w(\theta, \langle x, \theta \rangle) dx d\theta \end{aligned} \tag{2.38}$$

In the inner integral, θ is fixed so we can use corollary 2.33 to the hyperplane integration theorem to obtain

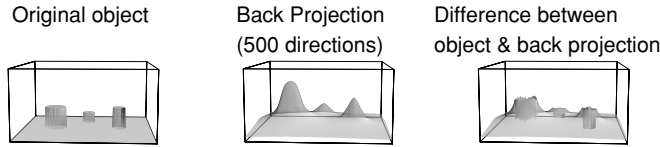
$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) w(\theta, \langle x, \theta \rangle) dx &= \int_{p \in \mathbb{R}} \int_{\langle \theta, x \rangle = p} f(x) w(\theta, \langle x, \theta \rangle) dx dp \\
 &= \int_{p \in \mathbb{R}} \int_{\langle \theta, x \rangle = p} f(x) w(\theta, p) dx dp \\
 &= \int_{p \in \mathbb{R}} w(\theta, p) \int_{\langle \theta, x \rangle = p} f(x) dx dp \\
 &= \int_{p \in \mathbb{R}} w(\theta, p) Rf(\theta, p) dp
 \end{aligned}$$

Substituting this result back into equation (2.38) we get

$$\begin{aligned}
 \langle f, R^\# w \rangle &= \int_{S^{n-1}} \int_{p \in \mathbb{R}} w(\theta, p) Rf(\theta, p) dp d\theta \\
 &= \langle Rf, w \rangle
 \end{aligned}$$

■

In chapter 1 we investigated the direct use of backprojection for the inversion of the Radon transform. Recall that even for a simple object consisting of the characteristic function of three disks, the direct use of backprojection gave very poor results, as indicated in the accompanying figure:



The reason for this poor result is indicated by the following result.

Theorem 2.76 (The Backprojection Theorem). *If f is a nonnegative Lebesgue measurable function on $\mathbb{R}^n$ then we have*

$$R^\# Rf = T * f$$

where the function T is defined by

$$T(x) = |S^{n-2}| \frac{1}{|x|}$$

Recall that $|S^{n-2}|$ denotes the $n - 2$ -dimensional measure of the unit $n - 2$ sphere.

The proof of this theorem depends on some concepts beyond the prerequisites for this chapter, so we refer the interested reader to chapter 3, theorem 3.67, for a more general

result or to Natterer [444]. The convolution $T * f$ differs only by a multiplicative constant from an important operator called the Riesz potential which is studied in detail in chapter 3.

Remark 2.77. Lebesgue integration is very powerful, although there are some integrals that would be useful, but which lie outside the scope of Lebesgue integration. For example $\int_{-1}^1 \frac{1}{x} dx$ is not Lebesgue integrable. However we can assign a meaning to this integral via the equation

$$\int_{-1}^1 \frac{1}{x} dx = \lim_{\varepsilon \rightarrow 0} \int_{[-1, -\varepsilon] \cup [\varepsilon, 1]} \frac{1}{x} dx \quad (2.39)$$

In this simple example, the limit evaluates to zero. However, many interesting examples of this type of integration exist. Integrals of the form (2.39) are called Cauchy principal value integrals. They are sometimes denoted by the symbol $(p.v.) \int_A f(x) dx$, but if it is obvious that the singularity is being treated by the Cauchy method, then we usually drop the prefix $(p.v.)$. Cauchy principal value integrals are a subset of the more general class of singular integrals.

The integral defining the convolution $T * f$ is, in general, a singular integral. However, we will show in chapter 3 that if the function $(1 + |x|)^{-1} f(x)$ is integrable, then the integral defining the convolution $T * f(x)$ exists as a finite Lebesgue integral for almost all x (compare, theorem 3.64, chapter 3). In Chapter 3 we also show that the condition that $(1 + |x|)^{-1} f(x)$ is integrable is essentially the most general condition under which the integral defining the convolution $T * f(x)$ is not a singular integral (see chapter 3, theorem 3.65). Furthermore, $(1 + |x|)^{-1} f(x)$ is integrable if, and essentially only if, the Radon transform of f exists almost everywhere (see chapter 3, theorem 3.35).

Remark 2.78. This remark requires some more advanced knowledge of Fourier transforms.

The function T is not integrable, although it is a tempered function. Readers who are conversant with tempered distributions will know that any tempered function has a Fourier transform that is a tempered distribution. In the case of T it is possible to show that its Fourier transform is also a tempered function: $\widehat{T}(\xi) = 2^{\frac{n}{2}} \pi^{\frac{n}{2}-1} \frac{1}{|\xi|^{n-1}}$.

The convolution of integrable functions is again integrable. Therefore, if g and f are integrable functions, then $g * f$ has a Fourier transform and the convolution theorem for the Fourier transform gives the following formula:

$$(g * f)^{\wedge}(\xi) = (2\pi)^{\frac{n}{2}} \widehat{g}(\xi) \widehat{f}(\xi).$$

However, if we replace g by the nonintegrable function T , then the formula $(T * f)^{\wedge}(\xi) = (2\pi)^{\frac{n}{2}} \widehat{T}(\xi) \widehat{f}(\xi) = 2^n \pi^{n-1} |\xi|^{1-n} \widehat{f}(\xi)$, is not true for all integrable functions f . The study of which functions do give rise to this formula and the consequences for inverting the Radon transform are pursued in chapter 3, section 3.8. However, we do

indicate later in this chapter how one may be led to an exact inversion formula for the Radon transform via the backprojection theorem.

The next result, theorem 2.79, is the mathematical foundation of the filtered backprojection method which was proposed in chapter 1 to remedy the deficiency of the backprojection method. Theorem 2.79 rigorously supports the experimental evidence in chapter 1 that applying backprojection to a suitably filtered Radon transform will yield a good reconstruction of the original function.

Theorem 2.79 (Filtered Backprojection). *Let f be integrable on $\mathbb{R}^n$ and let w be a bounded measurable function on the cylinder $Z^n = S^{n-1} \times \mathbb{R}$. Then we have*

$$R^\#(w) * f = R^\#(w * Rf) \quad (2.40)$$

Proof. Use the definition of convolution on $\mathbb{R}^n$, the definition of backprojection, and the extended Fubini theorem to obtain

$$\begin{aligned} (R^\#(w) * f)(x) &= \int_{\mathbb{R}^n} (R^\#(w)(x - y)) f(y) dy \\ &= \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} w(\theta, \langle x - y, \theta \rangle) d\theta \right) f(y) dy \\ &= \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} w(\theta, \langle x, \theta \rangle - \langle y, \theta \rangle) f(y) dy \right) d\theta \end{aligned} \quad (2.41)$$

The use of the Fubini theorem is justified because w is a bounded measurable function on S^{n-1} and f is an integrable function on $\mathbb{R}^n$ so the function

$$w(\theta, (\langle x, \theta \rangle - \langle y, \theta \rangle)) f(y) \quad (2.42)$$

is integrable on the cylinder $Z^n = S^{n-1} \times \mathbb{R}$.

Now fix $x \in \mathbb{R}^n$ and $\theta \in S^{n-1}$ and define the function h on $\mathbb{R}^n$ by

$$h(y) = w(\theta, (\langle x, \theta \rangle - \langle y, \theta \rangle)) f(y)$$

The function h is the restriction of the function defined in (2.42) to the set where both x and θ are fixed constants. As in the case of (2.42), h is an integrable function on the cylinder $\mathbb{R}^n$. Therefore by corollary 2.33 we have

$$\int_{\mathbb{R}} R_\theta h(s) ds = \int_{\mathbb{R}^n} h(y) dy \quad (2.43)$$

But $R_\theta h(s)$ is the integral of h over the hyperplane $\langle y, \theta \rangle = s$. On this hyperplane we have $w(\theta, (\langle x, \theta \rangle - \langle y, \theta \rangle)) = w(\theta, (\langle x, \theta \rangle - s))$ which is independent of y so

$$\begin{aligned} R_\theta h(s) &= \int_{\langle y, \theta \rangle = s} h(y) dy = \int_{\langle y, \theta \rangle = s} w(\theta, (\langle x, \theta \rangle - \langle y, \theta \rangle)) f(y) dy \\ &= w(\theta, (\langle x, \theta \rangle - s)) \int_{\langle y, \theta \rangle = s} f(y) dy \\ &= w(\theta, (\langle x, \theta \rangle - s)) Rf(\theta, s) \end{aligned}$$

Thus, from equation (2.43) we get

$$\begin{aligned} \int_{y \in \mathbb{R}^n} w(\theta, (\langle x, \theta \rangle - \langle y, \theta \rangle)) f(y) dy \\ = \int_{\mathbb{R}} w(\theta, (\langle x, \theta \rangle - s)) Rf(\theta, s) ds \end{aligned} \quad (2.44)$$

which is valid for any (θ, x) in the cylinder Z^n . Substituting (2.44) into (2.41) and taking into account the definition of backprojection and the definition of convolution we get

$$\begin{aligned} (R^\#(w) * f)(x) &= \int_{S^{n-1}} \left(\int_{\mathbb{R}} w(\theta, (\langle x, \theta \rangle - s)) Rf(\theta, s) ds \right) d\theta \\ &= \int_{S^{n-1}} (w * Rf)(\theta, \langle x, \theta \rangle) d\theta \\ &= R^\#(w * Rf)(x) \end{aligned}$$

which proves the theorem. ■

Equation (2.40) is the basis of the filtered backprojection method for approximately reconstructing a function from its Radon transform.

Define $A = R^\#(w)$. Then A is called a **filter** or **point spread function** and w is called a **reconstruction kernel**. This is the reason the name “filtered backprojection” is given to this procedure. If we can establish a family of such filters A which form an approximate identity, then the approximate identity lemma shows that, in the limit, $A * f$ approaches the function f . Therefore, we can choose a specific element A of the approximate identity such that $A * f$ approximates f to any desired degree of accuracy.

Equation (2.40) becomes

$$A * f = R^\#(w * Rf)$$

and we can now think of the left-hand side, $A * f$, as an approximation to the unknown function f . However, both Rf (obtained from the x-ray data) and w are known.

Therefore, the approximation to the unknown function f may be computed from the known data via this equation.

Let α be a general procedure which operates on a function f and produces an approximation $\alpha(f)$ to f . Let us consider this idea when α represents some approximate, linear tomographic reconstruction algorithm. Paraphrasing a remark of K. T. Smith ([562], page 12) we note that if the approximation procedure results in the same reconstruction in Cincinnati and San Francisco, then it must be translation invariant. Since it is reasonable to assume that a CT scanner works the same in both Cincinnati and San Francisco, we see that α is a linear and translation invariant operator. It is known that linear and translation invariant operators must be of the form $\alpha(f) = A * f$. The precise statement of the theorem connecting linear translation invariant operators to convolutions may be found in Stein and Weiss [583], theorem 3.16. Then theorem 2.79 shows that as long as $A = R^\#(w)$ for some w defined on the cylinder, then the only type of linear, translation invariant reconstruction process that we could possibly obtain from the Radon transform is the filtered backprojection method.

In practice the kernel w can be chosen to be a bounded continuous function such that when its backprojection A is convolved with f , then $A * f$ does give a good approximation to f .

Two common kernels (actually families of kernels) are the Ramachandran-Lakshminarayanan kernel, w_{LR} [510], given by

$$w_{LR}(\theta, s) = \frac{b^2}{4\pi^2} \left(\frac{\cos(bs) - 1}{b^2 s^2} + \frac{\sin(bs)}{bs} \right)$$

(with the value at 0 being $\frac{1}{8} \frac{b^2}{\pi^2}$ to preserve continuity) and the Shepp-Logan kernel, w_{SL} [557],

$$w_{SL}(\theta, s) = \frac{b^2}{2\pi^3} \frac{\frac{\pi}{2} - bs \sin(bs)}{\left(\frac{\pi}{2}\right)^2 - b^2 s^2}$$

(with the value at $\pm \frac{\pi}{2b}$ being $\frac{b^2}{2\pi^4}$). Natterer's book [444], chapter 5, justifies the fact that a proper choice of filter together with an appropriate choice of the parameter b does allow the filtered backprojection method to give a good approximate inversion.

Nievergelt [449], (compare, Nievergelt [450]) gives a very elementary derivation of the filtered backprojection process on $\mathbb{R}^2$ for the special case where the kernel w_b is defined on $S^1 \times \mathbb{R}$ by

$$w_b(\theta, s) = \begin{cases} \frac{1}{\pi b^2} & \text{if } |s| \leq b \\ \frac{1}{\pi b^2} \left(1 - \frac{|s|}{\sqrt{s^2 - b^2}} \right) & \text{if } |s| > b \end{cases}$$

He is able to prove that

$$R^\# w_b(x) = \begin{cases} \frac{1}{\pi b^2} & \text{if } |x| \leq b \\ 0 & \text{if } |x| > b \end{cases}$$

Now recall remark 2.73 about the extension of approximate identities to $\mathbb{R}^n$. It is easy to check that if we let $\varphi_b = R^\# w_b$ and if we let $\varphi = \varphi_1$, then

$$\varphi_b(x) = \frac{1}{b^2} \varphi\left(\frac{x}{b}\right)$$

and hence the family φ_b is an approximate identity on $\mathbb{R}^2$. The filtered backprojection theorem then yields

$$\begin{aligned} f &= \lim_{b \rightarrow 0} f * \varphi_b \\ &= \lim_{b \rightarrow 0} R^\#(w_b) * f \\ &= \lim_{b \rightarrow 0} R^\#(w_b * Rf) \end{aligned}$$

which provides both an inversion formula and an approximate inversion formula for the Radon transform. Nievergelt's kernel is not used in practical computerized tomography, probably because of engineering concerns in which the Ramachandran-Lakshminarayanan and Shepp-Logan kernels give better performance under noisy conditions. These kernels are designed to eliminate or taper off high frequencies in the unknown function. The most obvious such approach is to accept all frequencies in a given bandwidth and to reject higher frequencies. This gives rise to the ideal low-bandpass filter which is the kernel used by Nievergelt. Despite its forbidding name, the ideal low bandpass filter is a very simple concept mathematically. It merely multiplies the Fourier transform by the characteristic function of a disk. The radius of this disk is the bandwidth B . Therefore, the ideal low-bandpass filter cuts off any frequencies greater than B , but allows lower frequencies to pass through unchanged. Although the ideal low-bandpass filter is the simplest filter from the mathematical point of view, it does not give the best performance relative to noise reduction. The Ramachandran-Lakshminarayanan and Shepp-Logan kernels are designed to give better reconstructions than the ideal low-bandpass filter.

On the other hand, the Ramachandran-Lakshminarayanan, Shepp-Logan and the ideal low-bandpass filter kernels are not approximate identities. If w represents any of these kernels and if we define $\varphi = R^\# w$, then it can be shown that

$$\varphi\left(\frac{x}{b}\right) = \sqrt{b} \frac{1}{b^n} \varphi_1\left(\frac{|x|}{b}\right)$$

so we are off by a factor of $\sqrt{b}$ from having an approximate identity. Therefore, the kernels based on restricting bandwidth are appropriate for approximate reconstructions but not exact inversion.

We can also use the backprojection theorem in conjunction with remark 2.78 to obtain an exact inversion formula for the Radon transform on $\mathbb{R}^n$.

We mentioned in Remark 2.78 that the Fourier transform convolution theorem does not generally apply to the convolution $T * f$. However, there is a large class of functions

for which this convolution theorem is true. For those functions we will have

$$(T * f)^\wedge(\xi) = 2^n \pi^{n-1} \frac{1}{|\xi|^{n-1}} \widehat{f}(\xi). \quad (2.45)$$

The last ingredient in the proposed inversion theorem is the operator Λ defined by letting Λg be the inverse Fourier transform of $|\xi| \widehat{g}(\xi)$. Composing Λ ($n - 1$) times gives Λ^{n-1} with the property

$$\left(\Lambda^{n-1} g \right)^\wedge = |\xi|^{n-1} \widehat{g}(\xi).$$

Theorem 2.80. *If f is a function satisfying equation (2.45), then we have the following inversion formula for the Radon transform:*

$$\frac{1}{2^n \pi^{n-1}} \Lambda^{n-1} R^\# R f = f$$

The class of functions for which this is valid includes rapidly decreasing differentiable functions and compactly supported integrable functions. These functions include all the functions of practical interest in tomography.

Proof. We start by taking the Fourier transform of $\Lambda^{n-1}(R^\# R f)$. From the various definitions we then have

$$\begin{aligned} \left(\Lambda^{n-1} (R^\# R f) \right)^\wedge &= |\xi|^{n-1} \left(R^\# R f \right)^\wedge(\xi) \\ &= |\xi|^{n-1} (T * f)^\wedge(\xi) \end{aligned}$$

The last step comes from the backprojection theorem: $R^\# R f = T * f$. Continuing the calculation by using equation (2.45) we get

$$\begin{aligned} \left(\Lambda^{n-1} (R^\# R f) \right)^\wedge &= |\xi|^{n-1} \left(2^n \pi^{n-1} \frac{1}{|\xi|^{n-1}} \widehat{f}(\xi) \right) \\ &= 2^n \pi^{n-1} \widehat{f}(\xi) \end{aligned}$$

Taking the inverse Fourier transform yields the desired inversion formula. ■

Of course, we need to describe the class of functions for which this formula is valid and to justify the convolution theorem in equation (2.45). This is done, in greater generality, in chapter 3, sections 3.6, 3.7, and 3.8. In particular, see corollary 3.80.

Practical application of the filtered backprojection method is something of an art. An appropriate choice of the filter along with the parameter b and the (necessarily finite) number of directions is crucial to getting a useful reconstruction. A discussion of this topic is beyond the scope of this book. However, the books by Herman [296], Kak and Slaney [328], Natterer [444], and Natterer and Wübbeling [446] are useful in this regard as are the papers by Bracewell and Riddle [65], Chang and Herman [85], Lewitt, Bates, and Peters [389], Ramachandran and Lakshminarayanan [510], Shepp and Logan [557], Smith [562, 563], and Smith and Keinert [564].

2.8 Inversion of the Radon Transform in $\mathbb{R}^2$ – Radon's Proof

Having shown the approximate inversion of the Radon transform in the preceding section, we now present Radon's method for inverting the Radon transform in $\mathbb{R}^2$. This is one of several major results presented by Johann Radon in his ground-breaking paper of 1917 [508]. The reader who is interested in reading the original may find it convenient to look in the following sources which have reprints of his paper: *75 Years of Radon Transform* [211] or Helgason [277].⁷ These reprints are in the original German. Furthermore, Deans [124] and Parks [509] have English translations of the paper. The translation by Parks also has corrections to misprints in the original paper.

More general inversion results are given in Chapter 3.

Before proceeding to the proof let us remark that Radon observed that an inversion formula could be developed very quickly via the use of Abel integral equations. Radon was not convinced of the rigor of this approach, but Gel'fand, Gindikin, and Graev [189] have a modern, rigorous, and beautifully concise development by using this approach. I highly recommend the proof in [189] to the more advanced reader. The approach here is longer to respect the more modest prerequisites assumed for this chapter.

Radon defines for every point $P = (x, y)$ and for functions f defined on $\mathbb{R}^2$:

$$\bar{f}_P(r) = \frac{1}{2\pi} \int_0^{2\pi} f(x + r \cos \phi, y + r \sin \phi) d\phi, \quad (2.46)$$

$$F(p, \phi) = \int_{-\infty}^{\infty} f(p \cos \phi - s \sin \phi, p \sin \phi + s \cos \phi) ds \quad (2.47)$$

$$\bar{F}_P(q) = \frac{1}{2\pi} \int_0^{2\pi} F(x \cos \phi + y \sin \phi + q, \phi) d\phi \quad (2.48)$$

In current terminology $F(p, \phi)$ is called the Radon transform, namely, $F(p, \phi) = Rf(\theta, p)$, where $\theta = (\cos \phi, \sin \phi)$.

The operators $\bar{f}_P$ and $\bar{F}_P$ are mean value operators on the original function and its Radon transform, respectively. Note that these are actually functions of (x, y) as well as the stated variable. The line $\langle \theta, x \rangle = p$ is a tangent line to the circle of radius p centered at the origin. As ϕ varies, $F(p, \phi)$ runs through the line integrals on all tangent lines to the circle of radius p centered at the origin. More generally one can show that the set of lines defined by the equation

$$x \cos \phi + y \sin \phi = x_0 \cos \phi + y_0 \sin \phi + q$$

generates the set of tangent lines to the circle of radius q centered at (x_0, y_0) as ϕ ranges from 0 to 2π . Therefore the mean value function $\bar{F}_P(q) = \bar{F}_P(q, x, y)$ is obtained by integrating the Radon transform over all lines tangent to the circle of radius q centered at (x, y) . This is a dual operation to the Radon transform: in one case we integrate

⁷ Only the first edition [277], not the second edition [291] of Helgason's book *The Radon Transform* has the reprint of Radon's 1917 paper.

tangent lines to produce a value at a point, whereas in the other case we integrate over points to produce the value at a tangent line. There is another aspect to this duality. In section 2.7 we defined the backprojection operator as the formal adjoint of the Radon transform. We note here that, in terms of the backprojection operator, $\bar{F}_P(q)$ is the backprojection of f translated by $-q$ in the first variable.

Radon made the following assumptions about the functions f that he dealt with:

(a₁) f is continuous on $\mathbb{R}^2$.

(b₁) $\int_{\mathbb{R}^2} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} dx dy < \infty$

(c₁) $\lim_{r \rightarrow \infty} \bar{F}_P(r) = 0$

Remark 2.81. The conditions (a₁)–(c₁) are somewhat artificial, although they are certainly fulfilled by compactly supported⁸ continuous functions. More generally, a continuous function which is $O(|x|^{-N})$ with $N > 2$ will satisfy these conditions.⁹ This growth condition is the hypothesis of an inversion formula presented by Helgason for infinitely differentiable functions (Helgason [291], theorems 3.1 and 6.2). We have a generalization to measurable functions which are bounded near the origin and which satisfy this growth condition (chapter 3, corollary 3.77).

Functions which are $O(|x|^{-N})$ with $N > 2$ are integrable. There are functions that fail to be integrable yet which satisfy Radon's criteria: let $f(x) = \frac{1}{|x|^{\frac{3}{2}}}$ if $|x| \geq 1$ with $f(x) = 1$ elsewhere. Then f is easily shown to be a continuous function which is not integrable, yet which satisfies Radon's conditions (a₁)–(c₁) and hence has a Radon transform that can be inverted by the methods of this section.

In chapter 3 we derive a very general inversion formula for the Radon transform on a class of function that contains nonintegrable functions. The main property of functions in this class is that the Riesz potential of the order $n - 1$ exists almost everywhere. It is interesting to note that Radon's condition (b₁) for a continuous function f is equivalent to f having a Riesz potential of the order $n - 1$, which is finite almost everywhere. The interested reader can look at chapter 3, sections 3.7 and 3.8, especially theorems 3.64 and 3.65. The advanced reader will find an interesting update and generalization of Radon's theorem in Madych, Radon's inversion formulas, Trans, AMS, 356 (2004) 4475–4491.

Radon stated and presented proofs of the following theorems.

Theorem I. $F(p, \phi)$ is defined almost everywhere in the sense that for any given circle in the plane, those points on the circle on whose tangent lines the integral defining F does not exist form a set of measure zero on the circle.

Theorem II. The integral defining $\bar{F}_P(q)$ is absolutely convergent for all P and q .

⁸ A function f defined on $\mathbb{R}^n$ is said to be **compactly supported** if the set of points where $f(x) \neq 0$ is contained in a bounded region of $\mathbb{R}^n$.

⁹ The “big O” notation is defined as follows: we say that $f(x) = O(|x|^N)$ if there is a constant C such that $|f(x)| \leq C|x|^N$ for all $x \in \mathbb{R}^n$.

Theorem III. *The value of f is uniquely determined by $\overline{F}_P$ and can be found by the following computation:*

$$f(P) = -\frac{1}{\pi} \int_0^\infty \frac{d\overline{F}_P(q)}{q} \quad (2.49)$$

This Stieltjes integral can also be expressed in the form

$$f(P) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{\overline{F}_P(\varepsilon)}{\varepsilon} - \int_\varepsilon^\infty \frac{\overline{F}_P(q)}{q^2} dq \right) \quad (2.49')$$

We will now present proofs of theorems I to III. The presentation follows Radon's paper very closely, although we give more details.

Radon observes but does not prove that the double integral

$$\int_{x^2+y^2 > q^2} \frac{f(x, y)}{\sqrt{x^2 + y^2 - q^2}} dx dy \quad (2.50)$$

is absolutely convergent. The proof can be done by choosing any real number a larger than $|q|$ and breaking the integral into two pieces, the first I_1 with domain $q^2 < x^2 + y^2 < a^2$ and the second, I_2 , with domain $x^2 + y^2 \geq a^2$.

Noting that $\sqrt{\frac{r^2}{r^2 - q^2}}$ is a decreasing function for $r > q$, it is easy to see that there is a constant such that

$$\frac{1}{\sqrt{r^2 - q^2}} \leq K \frac{1}{r}$$

for $r \geq a$ (just take $K = \sqrt{\frac{a^2}{a^2 - q^2}}$). Then letting $r = \sqrt{x^2 + y^2}$ and substituting in the integral I_2 we get

$$\begin{aligned} |I_2| &\leq \int_{x^2+y^2 \geq a^2} \frac{|f(x, y)|}{\sqrt{x^2 + y^2 - q^2}} dx dy \\ &\leq K \int_{x^2+y^2 \geq a^2} \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} dx dy < \infty \end{aligned}$$

by condition (b_1) .

In the integral I_1 change to polar coordinates:

$$|I_1| \leq \int_0^{2\pi} \int_q^a \frac{|f(r \cos \phi, r \sin \phi)|}{\sqrt{r^2 - q^2}} r dr d\phi$$

After the change of variable $u^2 = r^2 - q^2$ we obtain

$$|I_1| \leq \int_0^{2\pi} \int_0^{\sqrt{a^2 - q^2}} \left| f\left(\sqrt{u^2 + q^2} \cos \phi, \sqrt{u^2 + q^2} \sin \phi\right) \right| du d\phi$$

which is finite because of the continuity of f . This proves the desired absolute convergence of the integral (2.50).

We now make the change of coordinates, from x, y to s, ϕ variables ($s \geq 0, 0 \leq \phi \leq 2\pi$):

$$x = q \cos \phi - s \sin \phi$$

$$y = q \sin \phi + s \cos \phi$$

in the integral (2.50). The Jacobian is

$$\left| \det \begin{bmatrix} -\sin \phi & -q \sin \phi - s \cos \phi \\ \cos \phi & q \cos \phi - s \sin \phi \end{bmatrix} \right| = s$$

(recall that $s \geq 0$) and we note that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} q \\ s \end{bmatrix}$$

Because the matrix is orthogonal we see that $x^2 + y^2 = q^2 + s^2$ so the integral (2.50) transforms to

$$\int_0^{2\pi} \int_0^\infty f(q \cos \phi - s \sin \phi, q \sin \phi + s \cos \phi) ds d\phi \quad (2.51)$$

Now replace ϕ by $\phi + \pi$ and s by $-s$. We can also replace q by $-q$ since the integral (2.50) depends only on q^2 . Then equation (2.51) transforms to

$$\int_0^{2\pi} \int_{-\infty}^0 f(q \cos \phi - s \sin \phi, q \sin \phi + s \cos \phi) ds d\phi$$

and adding these results and dividing by 2 gives

$$\begin{aligned} & \int_{x^2 + y^2 > q^2} \frac{f(x, y)}{\sqrt{x^2 + y^2 - q^2}} dx dy \\ &= \frac{1}{2} \int_0^{2\pi} \int_{-\infty}^\infty f(q \cos \phi - s \sin \phi, q \sin \phi + s \cos \phi) ds d\phi \end{aligned} \quad (2.52)$$

We note that the inner integral is now $F(q, \phi)$ so we get the following backprojection type of result:

$$\begin{aligned} \int_{x^2+y^2>q^2} \frac{f(x, y)}{\sqrt{x^2+y^2-q^2}} dx dy &= \frac{1}{2} \int_0^{2\pi} F(q, \phi) d\phi \\ &= \pi \bar{F}_0(q) \end{aligned} \quad (2.53)$$

(the last step following from the definition of the mean value function $\bar{F}_P$ when P is the origin).

From Fubini's theorem and the convergence of the integral (2.52) we see that the inner integral in (2.52), which is $F(q, \phi)$, exists for almost all ϕ which proves theorem I. Theorem II is proved similarly.

We now give Radon's proof of the identity (2.49') in theorem III.

Introduce polar coordinates in equation (2.50) and use the definition of $\bar{f}_0(r)$ to get

$$\begin{aligned} \bar{F}_0(q) &= \frac{1}{\pi} \int_q^\infty \int_0^{2\pi} \frac{f(r \cos \phi, r \sin \phi)}{\sqrt{r^2 - q^2}} r d\phi dr \\ &= \int_q^\infty 2\bar{f}_0(r) \frac{r}{\sqrt{r^2 - q^2}} dr \end{aligned} \quad (2.54)$$

We note that there is either a misprint or a mistake in Radon's paper at the point where this equation is stated (page 265, equation (2) of [508]).^{10, 11} However, Radon does use the correct form in the remainder of his proof of theorem III.

Radon notes that the conditions $(a_1)-(c_1)$ on the function f are invariant under rigid motions of the plane, so it suffices to prove equation (2.49') for $P = 0$. Using

¹⁰ Radon makes a change to polar coordinates in the following equation appearing on page 264 of his paper [508]

$$\int_{x^2+y^2>q^2} \frac{f(x, y)}{\sqrt{x^2+y^2-q^2}} dx dy$$

obtaining, at the bottom of the same page

$$\int_q^\infty dr \int_0^{2\pi} \frac{f(r \cos \phi, r \sin \phi)}{\sqrt{r^2 - q^2}} d\phi$$

This should be

$$\int_q^\infty r dr \int_0^{2\pi} \frac{f(r \cos \phi, r \sin \phi)}{\sqrt{r^2 - q^2}} d\phi$$

¹¹ In his translation [509] of Radon's paper, P. C. Parks has included several more corrections.

equation (2.54), the right-hand side of equation (2.49') becomes

$$\begin{aligned} & \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{\bar{F}_0(\varepsilon)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\bar{F}_0(q)}{q^2} dq \right) \\ &= \frac{2}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} \int_{\varepsilon}^{\infty} \frac{r \bar{f}_0(r)}{\sqrt{r^2 - \varepsilon^2}} dr - \int_{\varepsilon}^{\infty} \int_q^{\infty} \frac{r \bar{f}_0(r)}{\sqrt{r^2 - q^2}} dr \frac{1}{q^2} dq \right) \end{aligned} \quad (2.55)$$

At this point Radon suggests computing the iterated integral in reverse order and then, without any detail, states that the value of equation (2.55) becomes

$$\frac{2}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \frac{\bar{f}_0(r)}{r \sqrt{r^2 - \varepsilon^2}} dr$$

which expression yields, he claims, $\bar{f}_0(0) = f(0, 0)$, “wie unschwer zu zeigen ist” ([508], page 266). In fact, this expression is wrong and should be $\frac{2}{\pi} \varepsilon \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \frac{\bar{f}_0(r)}{r \sqrt{r^2 - \varepsilon^2}} dr$. It is probably better, therefore, to provide a few details.

First, it is easy to check, via a simple trigonometric substitution, that

$$\int_{\varepsilon}^r \frac{1}{q^2 \sqrt{r^2 - q^2}} dq = \frac{1}{r^2} \frac{\sqrt{r^2 - \varepsilon^2}}{\varepsilon}$$

We can use this relation to see that when the order is switched in the iterated integral in equation (2.55) we obtain:

$$\begin{aligned} & \int_{\varepsilon}^{\infty} r \bar{f}_0(r) \left(\int_{\varepsilon}^r \frac{1}{q^2 \sqrt{r^2 - q^2}} dq \right) dr \\ &= \int_{\varepsilon}^{\infty} r \bar{f}_0(r) \left(\frac{1}{r^2} \frac{\sqrt{r^2 - \varepsilon^2}}{\varepsilon} \right) dr \\ &= \frac{1}{\varepsilon} \int_{\varepsilon}^{\infty} \bar{f}_0(r) \frac{\sqrt{r^2 - \varepsilon^2}}{r} dr \end{aligned}$$

For $r > \varepsilon > 0$ we certainly have $r^2 - \varepsilon^2 \leq r^2$ and from this it follows easily that $\frac{\sqrt{r^2 - \varepsilon^2}}{r} \leq \frac{r}{\sqrt{r^2 - \varepsilon^2}}$. Hence we obtain the estimate

$$\int_{\varepsilon}^{\infty} |\bar{f}_0(r)| \frac{\sqrt{r^2 - \varepsilon^2}}{r} dr \leq \int_{\varepsilon}^{\infty} |\bar{f}_0(r)| \frac{r}{\sqrt{r^2 - \varepsilon^2}} dr = \left| \frac{1}{2} \bar{F}_0(\varepsilon) \right| < \infty$$

by equation (2.54) and theorem II. This shows, by Fubini's theorem, that the double integral associated to the iterated integral is absolutely convergent and that the change

in the order of integration is justified. Using this result back in equation (2.55) yields

$$\begin{aligned}
 & \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{\bar{F}_0(\varepsilon)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\bar{F}_0(q)}{q^2} dq \right) \\
 &= \frac{2}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} \int_{\varepsilon}^{\infty} \frac{r \bar{f}_0(r)}{\sqrt{r^2 - \varepsilon^2}} dr - \frac{1}{\varepsilon} \int_{\varepsilon}^{\infty} \bar{f}_0(r) \frac{\sqrt{r^2 - \varepsilon^2}}{r} dr \right) \\
 &= \frac{2}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\int_{\varepsilon}^{\infty} \bar{f}_0(r) \frac{\varepsilon}{r \sqrt{r^2 - \varepsilon^2}} dr \right) \tag{2.56}
 \end{aligned}$$

If we now define $\varphi_{\varepsilon}(r) = \frac{2}{\pi} \varepsilon \frac{1}{r \sqrt{r^2 - \varepsilon^2}}$ on $[\varepsilon, \infty)$, with $\varphi_{\varepsilon}(r) = 0$ elsewhere, we see that we can interpret the right-hand side of equation (2.49') as the convolution

$$\lim_{\varepsilon \rightarrow 0} \bar{f}_0 * \varphi_{\varepsilon}(0)$$

and this suggests applying the approximate identity theorem, lemma 2.70. We can indeed do this because if we define $\varphi(r) = \varphi_0(r)$ then it is easy to verify that:

1. φ satisfies the hypotheses of lemma 2.70.
2. $\varphi_{\varepsilon}(r) = \frac{1}{\varepsilon} \varphi\left(\frac{r}{\varepsilon}\right)$
3. $\bar{f}_0$ satisfies the conditions of lemma 2.70 because from condition (a_1) that f is continuous it follows easily that the mean value function $\bar{f}_0$ is also continuous. This in conjunction with condition (c_1) that $\lim_{r \rightarrow \infty} \bar{f}_0(r) = 0$ yields that $\bar{f}_0$ is also bounded.

Therefore, we can apply lemma 2.70 to conclude that

$$\bar{f}_0(0) = \lim_{\varepsilon \rightarrow 0} \bar{f}_0 * \varphi_{\varepsilon}(0)$$

Finally we observe, from equation (2.46), that

$$\begin{aligned}
 \bar{f}_0(0) &= \frac{1}{2\pi} \int_0^{2\pi} f(0, 0) d\phi \\
 &= f(0, 0)
 \end{aligned}$$

Putting together all these facts we see that we have established equation (2.49'):

$$f(P) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \left(\frac{\bar{F}_P(\varepsilon)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\bar{F}_P(q)}{q^2} dq \right) \tag{2.57}$$

We can now use this result to verify equation (2.49) in theorem III. Integration by parts in $\int_0^\infty \frac{d\bar{F}_P(q)}{q}$ yields

$$\lim_{q \rightarrow \infty} \frac{\bar{F}_P(q)}{q} - \lim_{\varepsilon \rightarrow 0} \frac{\bar{F}_P(\varepsilon)}{\varepsilon} + \int_0^\infty \frac{\bar{F}_P(q)}{q^2} dq$$

It turns out that $\lim_{q \rightarrow \infty} \frac{\bar{F}_P(q)}{q} = 0$ (the reader can refer to Radon's paper for the details about this point). Combining this result with equation (2.57) yields formula (2.49).

Remark 2.82. Radon did not use approximate identities explicitly. He basically stated that it is obvious that $\int_\varepsilon^\infty \frac{2}{\pi} \bar{f}_0(r) \frac{\varepsilon}{r\sqrt{r^2 - \varepsilon^2}} dr$ converges to $\bar{f}_0(0)$ as $\varepsilon \rightarrow 0$. But this is only obvious if you know about approximate identities and analyze this integral in the form of a convolution with an approximate identity, as we did previously.

2.9 Additional References and Results

The historical remarks, further references, and applications are of general interest. The section on further results is for readers with a very good background in advanced mathematics. Here, one may find many results about the Radon transform beyond the basic results presented in this chapter. References but not proofs are provided for these more advanced results.

2.9.1 Historical Remarks

According to Cormack [104], “the first person I know of who tackled Radon's problem was the great Dutch physicist H. A. Lorentz.” Lorentz actually solved the problem of inverting the three-dimensional Radon transform before Radon's 1917 paper [508]. Cormack went on to say “We have no idea why Lorentz thought of the problem, or what his method of proof was, and the only reason we know of his work is that the above result was attributed to him by Bockwinkel who used the result in a long paper on the propagation of light in biaxial crystals.” Since Bockwinkel's paper [56] appeared in 1906, it seems that the first mathematical result in tomography along with the first application far outdated Radon's paper of 1917.

Radon's pioneering 1917 paper [508] contained the seeds of many of the currently active areas of mathematical research in tomography. In the years before 1970, his results were either “rediscovered” or generalized by various researchers. Uhlenbeck [605] inverted the n -dimensional Radon transform for odd n in 1925. In 1927 Mader [411] succeeded in finding an inversion formula for the Radon transform in all dimensions (compare, John [322]). In 1936 Cramér and Wold [110] inverted the Radon transform in the context of mathematical statistics; x-ray projections are abstractly the same as marginal distributions. Also in 1936 Ambartsumian [17] “rediscovered” the Radon transform and gave an interesting application to stellar astronomy; see the discussion below. Also of interest during the period before 1970 are the papers of

John [323–326], Gel’fand, Graev, and Vilenkin [194], Helgason [269], and Ludwig [405]. After 1970, there was an explosion of papers in the field, both theoretical and practical.

Curiously, many practical implementations of tomography were done without any knowledge of Radon’s work. Apparently, it was not until 1964 that Radon’s contribution was acknowledged in the practical CT literature. An early practical implementation of tomography was developed in 1936 by the astronomer Ambartsumian [17]. He obtained the same inversion formula as Radon [508], using essentially the same method. Remarkably, he did this without knowledge of Radon’s paper, which he learned of two years after his publication. The Nobel prize for Physiology or Medicine was awarded jointly to G. N. Hounsfield and A. M. Cormack in 1979. Each of these laureates made separate and important contributions to the field of computerized tomography. Apparently, however, neither Hounsfield nor Cormack were aware of Radon’s contributions. According to Deans [124], Page 5, “. . . Marr (1982) notes that Pincus (1964) apparently was the first person to develop a reconstruction algorithm with knowledge of the available material in the mathematics literature including Radon’s 1917 paper.” We cannot quote from Marr’s paper directly because the symposium volume [434] containing his paper was never published. Unfortunately, many other interesting and valuable papers were contained in this unpublished volume. In our bibliography these references are Marr [426] and Pincus [494].

There are more details on the history of computerized tomography in Chapter 1, section 1.10.1.

One of the earliest applications of tomography and the earliest development of the filtered backprojection method both occurred in astronomy. These are the application of Ambartsumian [17], already mentioned, and the development of filtered backprojection by Bracewell and Riddle [65].

In his 1936 paper [17] Ambartsumian considered the following problem: it is relatively easy to compute how fast a star is receding from or approaching to the earth. Astronomers refer to this as the radial velocity of the star. However, it is not obvious how to compute the motion of the star in other than the radial direction. The astronomer and physicist A. S. Eddington proposed the next best thing. He challenged astronomers to find a method of computing the number of stars with a specific velocity in three dimensions. Ambartsumian paraphrased Eddington’s challenge in the following way: “is it possible to find the distribution function $\varphi(\xi, \eta, \delta)$ of the components of stellar space velocities in the solar neighborhood from radial velocities alone . . . ? [18]”¹²

Ambartsumian solved this problem, in both two and three dimensions, by showing that what-we-now-call the Radon transform of the distribution function $\varphi(\xi, \eta, \delta)$ of the components of stellar space velocities is expressible in terms of the radial velocities of the stars. As remarked above, he then obtained the same inversion formula as Radon [508] by using essentially the same method.

Medical applications of tomography reconstruct an object in ordinary Euclidean space. However, Ambartsumian’s idea shows that one can use tomography to reconstruct

¹² Actually $\varphi(\xi, \eta, \delta)$ is the probability density function of the sample space, consisting of the velocity vectors of observable stars in the solar neighborhood.

the velocity distribution in velocity space just from the knowledge of radial velocities. He published the result only for the distribution of stellar velocities, but one could obtain the distribution of velocities of any system of particles knowing only the radial velocities. The idea of using tomography in medical applications did not appear until later.

Bracewell and Riddle [65] gave the earliest implementation of the filtered back-projection method while using tomographic methods to reconstruct a microwave map of the sun. Their input data, corresponding to x-rays in medical applications, were derived from strip integrals of microwave activity on the sun. These strip integrals were obtained by using radio telescopes with long, narrow apertures. The projections derived from such a telescope vary by angle as the earth rotates or as the orientation of the telescope is changed. Thereby a sample of the values of the Radon transform is obtained.

2.9.2 Further References

The reader may want to consult the references already presented in chapter 1, section 1.10.2, some of which are repeated here.

We presented an application to computerized tomography in diagnostic radiology in chapter 1. There are many other practical applications of tomography: the books by Deans [124], Herman [296], and Natterer [444] contain numerous detailed descriptions of applications of tomography to science, medicine, and technology. Natterer's paper [441], which is in German, gives a quick overview of the process of tomography described in his book [444].

The book by Gel'fand, Gindikin, and Graev [189] is an almost breathtakingly beautiful introduction to the theoretical aspects of mathematical tomography, written by some of the pioneers in the field.

The following books give a technical and in-depth treatment of the Radon transform and tomography: Deans [124], Gel'fand and Shilov [195], Gel'fand, Graev, and Vilenkin [194] (compare, Gel'fand and Graev [192]), Ehrenpreis [149], Helgason [287, 291, 292] (compare, [277], the original edition of "The Radon Transform" [291] cited previously), Herman [296], Natterer, [444], Natterer and Wübbeling [446], Natterer and Faridani [445], Kak and Slaney [328], and Ramm and Katsevich [513].

The next set of references give either a short introduction or passing mention to the Radon transform and tomography. See *Encyclopedia of Mathematics* [612] (Vol. 9, Sto-Zyg, pp. 179–181). This is a translation of the Soviet "Mathematical Encyclopedia" and has an excellent short introduction to the Radon transform and tomography. Vilenkin et al. [611] introduce the Radon transform in chapter 8. Khavin [351] gives a one-paragraph treatment of the Radon transform, J. S. Walker [616] gives a succinct, somewhat specialized introduction to Radon transform in an appendix (pp. 380–400). Terras [597] defines the Radon transform early in the book (pp. 107–119) and has very succinct guided exercises in which the reader can develop Radon inversion, and volume two of Terras [598] has a brief mention of applications to partial differential equations. Dym and McKean [138] include a short introduction to the Fourier inversion method for the Radon transform. John in his book [325] on partial differential equations

gives a quick introduction to the definition and inversion of the Radon transform in a few exercises.

There are some excellent survey articles on the Radon transform. Shepp and Kruskal [556] have an elementary and fascinating introduction to both the applied and theoretical aspects of the Radon transform. The articles by Strichartz [585] and Zalcman [625] also are of great interest. The articles of Smith, Solmon, and Wagner [566] and Smith [562] are very nice overviews of both the mathematical and practical aspects of tomography. Gindikin [202] is a beautiful and concise introduction to the theory and history of the Radon transform. This paper relates Radon's 1917 paper to modern day mathematics, in particular, to integral geometry and tomography.

Hlawka [305–309] implements a program to apply analytical number theory to mathematical analysis. One of his many applications is to the Radon transform.

2.9.3 Further Results

For the purposes of this section we let $C_v^\infty(\mathbb{R}^n)$ denote the space of C^∞ functions that vanish at infinity. We provide $C_v^\infty(\mathbb{R}^n)$ with the usual Frechet space topology defined by the sup-norm on functions and derivatives. The topological dual space to $C_v^\infty(\mathbb{R}^n)$ is called the space of **integrable distributions**, is denoted by $\mathcal{D}'_{L^1}(\mathbb{R}^n)$, and was introduced by Schwartz [546]. This space is a proper subspace of the space of all distributions on $\mathbb{R}^n$. It contains all the common test functions (rapidly decreasing functions, compactly supported measurable functions), all integrable functions, and compactly supported distributions. It also contains all finite Borel measures; consequently, the range of the Radon transform is contained in $\mathcal{D}'_{L^1}(S^{n-1} \times \mathbb{R})$ (integrable distributions are defined on $S^{n-1} \times \mathbb{R}$ in a similar manner). However, $\mathcal{D}'_{L^1}$ does not contain all L^p functions for $p > 1$, although it does contain some nonintegrable functions that are integrable on almost all hyperplanes.

2.9.3.1 General Results on the Radon Transform

Hertle [298, 299] considers the Radon transform defined on the space of integrable distributions $\mathcal{D}'_{L^1}(\mathbb{R}^n)$.

Hertle [298] shows that a continuous linear operator on certain function spaces that behaves under rotations, dilations, and translations like the Fourier transform must be a constant multiple of the Fourier transform. More precisely, let $\mathcal{D}(\mathbb{R}^n)$ be the space of compactly supported C^∞ functions with the usual topology and let $\mathcal{D}'(\mathbb{R}^n)$, the space of distributions on $\mathbb{R}^n$, be the dual space of $\mathcal{D}(\mathbb{R}^n)$. Let φ represent an arbitrary element of $\mathcal{D}(\mathbb{R}^n)$, let $\delta > 0$ and let $a \in \mathbb{R}^n$. Let $G : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator. Then G is a constant multiple of the Fourier transform $\mathcal{F}$, if and only if G commutes with rotations in the obvious way, commutes with dilations in the sense that $G(\varphi(\delta x)) = \delta^{-n}(G\varphi)(\delta^{-1}x)$, and commutes with translations in the sense that $G(\varphi(x+a)) = e^{i\langle a, x \rangle}(G\varphi)(x)$.

In the same paper Hertle proves an analogous result for the Radon transform. The most obvious analogy would relate to continuous linear operators from $\mathcal{D}(\mathbb{R}^n)$ to

$\mathcal{D}'(S^{n-1} \times \mathbb{R})$ with the same behavior as the Radon transform under translations, rotations, and dilations (compare, propositions 2.41, 2.42, and 2.50). However, Hertle shows by a counterexample that there is a continuous operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(S^{n-1} \times \mathbb{R})$, which satisfies the invariance properties but which is not a constant multiple of the Radon transform. Therefore, some restriction to the range space is necessary. The appropriate range space is the space $\mathcal{D}'_{L^1}(S^{n-1} \times \mathbb{R})$ of integrable distributions on the cylinder $S^{n-1} \times \mathbb{R}$. Hertle shows that if S is a continuous linear operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'_{L^1}(S^{n-1} \times \mathbb{R})$, which satisfies the same invariance under translations, rotations, and dilations that the Radon transform does, then S is a constant multiple of the Radon transform.

Kurusa [370] presents a characterization of the Radon transform similar to Hertle's, but it avoids the condition on the range. Kurusa also has an analogous characterization of the boomerang transform B , which is defined by

$$Bf(x) = \int_{\{\theta \in S^{n-1} : \langle \theta, x \rangle \geq 0\}} f(\langle \theta, x \rangle \theta) d\theta.$$

The boomerang transform clearly is closely related to the backprojection operator $R^\#$; hence, there is a characterization of the formal adjoint of the Radon transform. The boomerang transform was introduced by Szabo [592] in connection with his study of Hilbert's fourth problem.

Madych and Nelson [413] have a similar result for the finite Radon transform on $\mathbb{R}^2$. The finite Radon transform consists of a vector of projections of the form $R_{\theta_j} f$ for a finite set of indices: $j = 1, \dots, m$. A convenient notation is to let $\Theta = \{\theta_1, \dots, \theta_m\}$ and to define $R_\Theta f = (R_{\theta_1} f, \dots, R_{\theta_m} f)$. Madych and Nelson restrict attention to the situation in which the functions f are compactly supported in some closed disk. Most analytic methods of computed tomography depend on applying a linear transformation A (often a convolution) to the finite projection data $R_\Theta f$ to arrive at an approximate reconstruction of f . Madych and Nelson show that if A is such a linear, continuous operator such that $A \circ R_\Theta$ maps bounded functions to bounded functions and such that $A \circ R_\Theta$ is rotation and translation invariant, then there is a radial polynomial p of degree no greater than $2(m-1)$ such that $A \circ R_\Theta$ is a convolution operator with the polynomial kernel p :

$$A \circ R_\Theta f = f * p$$

They develop representations of such polynomial kernels in terms of ridge functions, compare, Madych and Nelson [414–416].

It is possible to define the Radon transform on certain spaces of distributions by taking advantage of the duality proved in theorem 2.75 $\langle Rf, w \rangle = \langle f, R^\# w \rangle$. If u were a distribution on a space of test functions T , then one could define Ru by

$$\langle Ru, \varphi \rangle = \langle u, R^\# \varphi \rangle$$

provided that φ is an arbitrary function in a space of test functions with the property that $R^\# \varphi \in T$. For example, Helgason [291] defines the Radon transform of compactly

supported distributions, whereas Gel'fand, Graev, and Vilenkin [194] and Ludwig [405] define the Radon transform of tempered distributions. Hertle [299] is able to define the Radon transform on the space of integrable distributions $\mathcal{D}'_{L^1}(\mathbb{R}^n)$, which we described previously. Hertle shows that the Radon transform is a continuous linear operator from $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ to $\mathcal{D}'_{L^1}(S^{n-1} \times \mathbb{R})$. He also shows that $\mathcal{D}'_{L^1}(\mathbb{R}^n)$ is the maximal space of distributions for which the Radon transform extends uniquely. Hertle has many other interesting results in this paper. Some of them are discussed in the following subsections.

Peters [477] considers functions f that are the sum of a compactly supported C^∞ function with a linear combination of a finite number of characteristic functions of compact subsets with smooth $n - 1$ -dimensional boundaries. He shows that the set of discontinuities of such a function f is directly related to the set of discontinuities of certain derivatives of the backprojection of the Radon transform of f .

Hahn and Quinto [252] show how to measure the distance between Borel probability measures on $\mathbb{R}^n$ in terms of the distance between the Radon transforms of these measures. They obtain results relative to various metrics, including the Prokhorov metric and the metrics induced by various Sobolev norms.

2.9.3.2 Continuity of the Radon Transform and Its Inverse

Continuity of the inverse Radon transform is meant as sequential continuity.

The Radon transform is a bounded and, hence continuous, linear operator on $L^1(\mathbb{R}^n)$. This will be shown in chapter 3, corollary 3.25. However, the Radon transform is unbounded on $L^2(\mathbb{R}^n)$. But Hertle [299] shows that the dual Radon transform $R^\#$ is a continuous operator in the following situations:

$$\begin{aligned} R^\# : C^\infty(S^{n-1} \times \mathbb{R}) &\rightarrow C^\infty(\mathbb{R}^n) \\ R^\# : C^\infty_v(S^{n-1} \times \mathbb{R}) &\rightarrow C^\infty_v(\mathbb{R}^n). \end{aligned}$$

An immediate consequence is that the Radon transform is a continuous operator in the following situations:

$$\begin{aligned} R : \mathcal{E}'(\mathbb{R}^n) &\rightarrow \mathcal{E}'(S^{n-1} \times \mathbb{R}) \\ R : \mathcal{D}'_{L^1}(\mathbb{R}^n) &\rightarrow \mathcal{D}'_{L^1}(S^{n-1} \times \mathbb{R}). \end{aligned}$$

Continuity of the inverse Radon transform would be very desirable, because it would make inversion of the Radon transform a stable, or well-posed, problem. However, the inverse Radon transform is not always continuous. The continuity depends on the domain of functions chosen for the Radon transform. For example, the Radon transform is not bounded on $C^0(B)$, where B is any ball in $\mathbb{R}^n$. In fact, Hertle [299] shows that a bounded sequence of continuous functions supported on the unit ball exists, such that the Radon transform of this sequence converges to zero uniformly, but for which the sequence converges nowhere. Additionally, the sequence does not converge weakly in $L^1(\mathbb{R}^n)$. Therefore, the inverse Radon transform is not continuous on $L^1(S^{n-1} \times \mathbb{R})$.

However, if one considers the Radon transform between certain Sobolev spaces, then one can show that both the Radon transform and its inverse exist and are continuous. Define $L^2_\alpha(\mathbb{R}^n)$ to be the Sobolev space of order α consisting of all tempered distributions u defined on $\mathbb{R}^n$ with the property that $(1 + |\xi|^2)^{\frac{\alpha}{2}} \widehat{u}(\xi)$ is square integrable. A similar definition applies to $L^2_\alpha(S^{n-1} \times \mathbb{R})$, except that the Fourier transform is taken only with respect to the second variable. If W is an open subset of $\mathbb{R}^n$, then $L^2_\alpha(W)$ consists of all $u \in L^2_\alpha(\mathbb{R}^n)$, such that $\text{supp}(u)$ is contained in the closure of W . A similar definition applies to $L^2_\alpha(S^{n-1} \times W)$, where W is an open subset of $\mathbb{R}$.

Hertle [299] and Louis [396] proved that:

Theorem 2.83. *The Radon transform is a bicontinuous map from $L^2_\alpha(B(r)) \rightarrow L^2_{\alpha+\frac{n-1}{2}}(S^{n-1} \times [-r, r])$.*

This result was previously obtained, in the case $\alpha = 0$, by Natterer [436] and Smith, Solmon, and Wagner [566]. Similar results for Radon transforms defined on compact elliptic manifolds were obtained by Guillemin [244] and Strichartz [584].

In the case that α is a positive integer, then L^2_α consists of square integrable functions whose weak derivatives are also square integrable. In general a function in L^2_α is considered to be smooth of order α . Therefore, theorem 2.83 can be interpreted to mean that the Radon transform increases the smoothness of a function by $\frac{n-1}{2}$.

Finally, Hertle is able to show in [299] that, in the following cases, the Radon transform and its inverse exist and are continuous:

$$\begin{aligned} R : \mathcal{E}'(\mathbb{R}^n) &\rightarrow \mathcal{E}'(S^{n-1} \times \mathbb{R}) \\ R : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathcal{S}(S^{n-1} \times \mathbb{R}) \\ R : C_0^\infty(\mathbb{R}^n) &\rightarrow C_0^\infty(S^{n-1} \times \mathbb{R}) \end{aligned}$$

Note that in [299] the space $C_0^\infty(\mathbb{R}^n)$ of compactly supported, infinitely differentiable functions is denoted by $\mathcal{D}(\mathbb{R}^n)$, whereas the space $C_v^\infty(\mathbb{R}^n)$ of infinitely differentiable functions, which vanish at infinity, is denoted by $C_0^\infty(\mathbb{R}^n)$.

2.9.3.3 Inversion of the Radon Transform

The following inversion formula for the Radon transform is classical; compare, Helgason [291], Ludwig [405], also Radon [508]. We prove it in chapter 3 (see theorem 3.53).

Theorem 2.84. *Let $f \in \mathcal{S}$. Then*

1. *If n is odd, then*

$$f(x) = \frac{(-1)^{\frac{n-1}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

and

2. If n is even, then

$$f(x) = \frac{(-1)^{\frac{n-2}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} H \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

Here H denotes the Hilbert transform and $\frac{\partial^{n-1}}{\partial p^{n-1}}$ refers to differentiation in the second variable. Theorem 2.84 seems to apply only to smooth functions. However, Hertle [299] shows that theorem 2.84 is valid on $\mathcal{D}'_{L^1}$. He does this by observing that the inversion formula in theorem 2.84 can be written in the form $f = R^\# K R f$, where K is the operator derived from the formulas in theorem 2.84. For example, if n is even, then $K(u) = \frac{(-1)^{\frac{n-2}{2}}}{2^n \pi^{n-1}} H \frac{\partial^{n-1}}{\partial p^{n-1}}(u)$. But $\mathcal{S} \subset C_v^\infty$ so, by duality, he can extend K to a map from $\mathcal{D}'_{L^1}$ to $\mathcal{S}'$. He then shows that for $u \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$, we have $u = R^\# K R u$, thereby extending the inversion formula to certain nonsmooth functions.

Deans [122] obtains a single inversion formula for the Radon transform that holds in both even and odd dimensions. In [123] Deans gives a Radon inversion formula for functions that are linear combinations of terms of the form $G(r) S(\theta)$, in polar coordinates, where G is a Gegenbauer polynomial and S is a spherical harmonic.

Hertle [299] proves a version of the slice-projection theorem for distributions in $\mathcal{D}'_{L^1}$. He goes on to prove that a compactly supported L^1 function f is determined by its Radon projections $R_\theta f$ for θ varying in an arbitrary open subset of S^{n-1} . A related result is that a finite measure is determined by its values on half-spaces of the form $\theta \cdot x < p$ for θ varying in an arbitrary open subset of S^{n-1} .

Inversion of the Radon transform can be accomplished via the slice-projection theorem by using inverse Fourier transformation. Practical implementations of this process discretize the data and the Fourier transform. Natterer [439] analyzes this discretization in $\mathbb{R}^n$ and provides two algorithms for choosing the finite number of directions so as to give the least error asymptotically. This least error is understood to be relative to a Sobolev norm.

2.9.3.4 More Results

Agranovsky and Quinto [9] (compare, [7]) present an application of the spherical Radon transform to approximation theory and partial differential equations. The spherical Radon transform is defined as follows:

$$R_{\text{sph}} f(x, r) = \int_{S(x, r)} f$$

where $S(x, r)$ is the sphere of radius r centered at x . Hence, the spherical Radon transform integrates functions over spheres in the same way that the Radon transform integrates functions over hyperplanes. A set of injectivity for the spherical Radon transform is a set S such that $R_{\text{sph}} f(x, r) = 0, \forall x \in S, \forall r \in \mathbb{R}^+$ implies $f = 0$. The function space $\mathcal{L}(S)$ is defined as the span of translates of S of continuous radial

functions on $\mathbb{R}^n$. The Coxeter system $\sum_N$ of lines in $\mathbb{R}^2$ is simply the set of lines connecting the origin to a $2N$ th root of unity. In [9], Agranovsky and Quinto show:

Theorem A. *If $n = 2$, then $\mathcal{L}(S)$ is dense in $C^0(\mathbb{R}^2)$ if and only if S is not contained in any set of the form $\omega(\sum_N) \cup F$, where ω is a rigid motion of $\mathbb{R}^2$, $\sum_N$ is a Coxeter system, and F is a finite subset of $\mathbb{R}^2$.*

They also show [9]:

Theorem B. *If $n = 2$, then the conditions in theorem A are necessary and sufficient for S to be a set of injectivity for the spherical Radon transform in $\mathbb{R}^2$.*

Theorems A and B have some very interesting consequences. Theorem B leads directly to a uniqueness result for the Darboux partial differential equation. For the heat equation $\frac{\partial u}{\partial t} = c^2 \Delta u$ on $\mathbb{R}^2 \times [0, T]$, with initial value $u(x, 0) = f(x)$, the set $Z(f)$ denotes the set where the temperature u is zero for all times t in the interval $[0, T]$. Agranovsky and Quinto show that if f is compactly supported, then either the initial temperature is identically zero or the zero temperature set $Z(f)$ is the union of a rigid motion of a Coxeter system and an isolated set of points in $\mathbb{R}^2$. Consequently for compactly supported initial distributions, it is impossible to have temperature zero all the time on a nonlinear smooth curve unless the temperature is zero everywhere and for all time.

Another consequence of theorems A and B is a similar result for the nodal sets of the membrane equation in $\mathbb{R}^2$. A third consequence is that a compactly supported continuous function f on $\mathbb{R}^2$ is uniquely determined by the values of its Riesz potentials $R_\lambda * f$ on a set $S \subset \mathbb{R}^2$, for λ in an open interval (a, b) , $b \leq 2$ unless S is the union of a rigid motion of a Coxeter system and a finite subset of $\mathbb{R}^2$. See [9] for details (compare, [8]).

Of interest related to the approximation theorem A is the paper of Agranovsky, Berenstein, and Kuchment [6], which shows that the closure of the set of all continuous radial functions with centers at the points of a closed surface in $\mathbb{R}^n$ are complete in the space $L^q(\mathbb{R}^n)$, if and only if $q \geq 2n/(n+1)$.

Of interest related to theorem B are the papers of Agranovsky [4, 5] and Agranovsky, Volchikov, and Zalcman [12] who prove that a cone in $\mathbb{R}^n$ is a set of injectivity for the spherical Radon transform if and only if it is not contained in the zero set of a nontrivial homogeneous harmonic polynomial (compare, Volchikov [613, 614]).

In another paper, Agranovsky and Quinto [10] present an application of the Radon spherical transform to the geometry of stationary sets of the wave equation. We consider the following Cauchy problem for the wave equation in $\mathbb{R}^n$:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta u, \\ u(x, t) &= 0 \text{ for } t > 0 \\ \frac{\partial u}{\partial t}(x, t) &= f(x) \text{ for } t > 0. \end{aligned} \tag{2.58}$$

It can be shown that there is a unique solution on $\mathbb{R}^n$ to this Cauchy problem, if one extends u to be zero on the half-space $t < 0$. The **stationary set of a function** $f \in C^\infty(\mathbb{R}^n)$, relative to the Cauchy problem (2.58), is defined to be the set $S(f) = \{x \in \mathbb{R}^n : u(x, t) = 0 \text{ for all } t > 0\}$, where u is the solution to the Cauchy problem (2.58). The wave equation extends directly to distributions, and Agranovsky and Quinto [10] show how to extend the definition of the stationary set to distributions. An interesting property of stationary sets is that the energy of a solution to the wave equation is constant in time on any domain Ω that is bounded by a stationary set.

Agranovsky and Quinto [10] prove that if f is a nonzero distribution supported on a finite set in $\mathbb{R}^n$, then its stationary set with respect to the wave equation (2.58) satisfies the following conditions.

1. $S(f)$ is an algebraic variety in $\mathbb{R}^n$ and is contained in the zero set of a harmonic polynomial.
2. After a suitable translation, the stationary set is of the form $S(f) = S_0 \cup V$, where V is an algebraic variety of codimension greater than 1. Furthermore, if $S_0 \neq \emptyset$, then S_0 is a harmonic cone, which is a codimension 1 real algebraic variety.

Agranovsky and Quinto also give a more specific description of the geometry of a stationary set for which the reader is referred to reference [10].

A paper of related interest is Agranovsky and Quinto [11], which gives properties of the stationary sets for the wave equation on domains related to crystallographic groups.

2.9.4 Applications of the Radon Transform

Louis has two interesting papers [400, 401], both with the title “Medical Imaging: State of the Art and Future Development.” The first paper [400] is fairly elementary from the mathematical point of view but has interesting informal descriptions of the process of data collection and reconstruction of images for medical diagnosis. The second paper [401] is a detailed overview of both the process and the mathematics of various forms of medical tomography. It contains discussions of local tomography, the Radon transform, the attenuated Radon transform, the exponential Radon transform, cone beam tomography, the FDK algorithm, and Grangeat’s method. It also deals with magnetic resonance imaging, ultrasound tomography, diffuse tomography, impedance tomography, and limited-view tomography. The emphasis in [401] is practical with attention being paid to discretization, stability, conditioning, and development of practical algorithms for tomography. For the reader interested in the practical side of tomography, the articles [400, 401] also provide an extensive list of references dealing with applications to medical imaging.

Another medical application of the Radon transform is to radiation therapy and radiation dose planning. Radiation therapy is used to shrink or destroy tumors. Radiation dose planning is concerned with maximizing the radiation dose at the site of the tumor while minimizing it elsewhere. This can be done by placing an array of x-ray sources of varying intensities at various locations around the body. The process of designing

this array is called radiation dose planning. As we saw in chapter 1, one computes the value of the backprojection operator $R^\# g(x)$ at the point x by averaging the values of g over hyperplanes passing through x . In dimension $n = 2$ this corresponds to the idea of computing the combined effect of x-rays passing through the point x from sources outside the body: the function $g(\theta, p)$ gives the intensity of the x-ray beam along the line defined by θ and p , and $R^\# g(x)$ gives the combined intensity of the resulting x-rays at the point x . Therefore, the backprojection operator gives a mathematical description of the process of radiation therapy. In their articles [108, 109], Cormack and Quinto apply the backprojection operator to the problem of radiation dose planning. Also see Levine, Gregerson, and Urie [388].

There are many applications of the Radon transform besides computerized tomography, which was treated in chapter 1, and other medical applications, such as those found in Louis [400, 401]. Natterer [444] and Deans [124] contain many such references which the reader may want to consult. The following applications are only a small sample of the myriad applications of the Radon transform.

Sirr and Waddle [561] report on the use of CT scanners to investigate the interior structure of violins and other bowed instruments. In one case the source of a buzz in a 1742 violin made by Carcassi in Florence was diagnosed by a CT scan. The cause of the defect was some loose dried glue in an otherwise undetectable worm track.

Michel [431] uses the Radon transform to give an infinitesimal version in dimension n of Blaschke's conjecture as to whether the canonical Riemannian metric on real projective n -space is the only one for which all geodesics are closed and of the same length.

Brédimas [69] uses the inverse of the spherical Radon transform to solve inverse problems for potentials.

Cavaretta, Micchelli, and Sharma [80] apply the Radon transform to statistics. They characterize certain extensions of univariant interpolation operators to multivariate ones in terms of the inverse Radon transform.

Peters [475] uses the Radon transform to derive laws of large numbers for certain random variables.

Richards [518] uses the Radon transform to derive some results about density functions useful in multivariate statistics.

Mayer-Wolf [429] presents an application of the discontinuous nature of the inverse Radon transform to probability theory.

Chapman [86] and Carswell and Moon [77] have applications of the Radon transform to seismology. More details can be found in chapter 5, section 5.10.

Goncharov [217, 218, 219] and Gel'fand and Goncharov [196] use the Radon transform to solve a problem arising in electron microscopy.

Maass [408, 409] studies a transform related to the Radon transform. This transform integrates functions over hyperbolas and is useful in wideband radar.

Henkin and Shaninin [294, 295] use a variant of the Radon transform to study a problem in mathematical economics.

Izen [319, 320], attempts to reconstruct the index of refraction of a supersonic gas flow using a limited view three-dimensional parallel beam Radon transform.

Quinto [504, 505, 506] shows how to reconstruct a function by using the exterior Radon transform. This reconstructs a function from the knowledge of its x-ray projections outside a fixed disk or annulus. The second and third papers give a method for the nondestructive testing of rocket engine exit cones and rocket body gaskets (the title of the third paper [506] is “Computed Tomography and Rockets”).

Globevnik [213] uses a support theorem for a Radon transform that integrates over circles surrounding the origin to prove several interesting facts about harmonic and analytic functions in the complex plane. Among these results is the following Morera-type theorem: if f is C^∞ near the origin and if the integral of f is zero around every circle surrounding the origin, then f is analytic. Of related interest are Globevnik [214], Globevnik and Quinto [215], and Grinberg and Quinto [240].

Roerdink [522] reviews the state of the art of CT as of 1992, including cardiac MRI.

The book [203] edited by S. G. Gindikin is a collection of applications of Radon transforms ranging from time series analysis to diffraction tomography.

Gindikin, Reeds, and Shepp [212] apply the spherical Radon transform to study dipoles of crystals.

Raymer, Beck, and McAlister [514] use tomographic methods to reconstruct the amplitude and phase structure of a quasimonochromatic wave field in a plane normal to its propagation direction by using only intensity measurements and refractive optics as input data.

Man’ko [420–422], Mancini, Man’ko, and Tombesi [419], and Alieva and Barbé [15] apply tomography to the study of quantum systems.

Berenstein [41] discusses an application of tomography to plasmas in space.

Bertero [52] describes applications of the Radon transform to medical imaging and astronomy. The applications to astronomy include image reconstruction in the Hubble telescope and in the large binocular telescope.

Desbat and Mennessier [128] study a generalized Radon transform related to Doppler stellar imaging. They show that the kernel of this type of Radon transform is nontrivial and thereby certain surface temperature distributions of stars may be invisible by this technique.

Ciotti [95] uses tomographic methods to study collisionless stellar systems.

Marzetta and Shepp [427] show that an ellipse is the only type of compact, convex subset of $\mathbb{R}^2$ for which the Radon transform depends only on the slope of the lines. They then give an application to motion detection.

Louis and Quinto [402] use local tomography (see section 3.11) to reconstruct object boundaries in shallow water by using sonar data.

Tamasan [596] uses the attenuated Radon transform to reconstruct isotropic scattering for the stationary transport of particles in a source-free scattering medium,

Cerejeiras, Schaeben, and Sommen [81] apply Radon transforms to the study of texture analysis.

Matulka and Collins [428] “rediscover” the Radon transform and apply it to reconstruct the density of a jet of gas. This has applications to aerodynamics. The input data are provided by holographic interferograms of the jet.

The k -Plane Transform, the Radon–John Transform

3.1 Introduction

Let us define a k plane to be any translation of a k -dimensional subspace of $\mathbb{R}^n$. Therefore, a k plane has the form $\eta + x$, where η is a k -dimensional subspace and $x \in \mathbb{R}^n$. Note that a hyperplane is therefore an $(n - 1)$ plane.

The Radon transform can be generalized so that the integration is performed on k planes instead of hyperplanes. The related transform is called a *k -dimensional Radon transform* or a *k -plane transform*. Some authors use the term *Radon–John transform*. We use the terms synonymously, and in this chapter we develop the theory of these transforms.

The main part of this chapter begins in section 3.3 with an investigation of the set of all k -dimensional linear subspaces of $\mathbb{R}^n$. This set is called the Grassmannian and is denoted by $G_{k,n}$. Grassmannians are not only sets, but they are also manifolds and measure spaces. We do not require the manifold structure, but we do need to know how to define a suitable measure on Grassmannians. This is done by introducing homogeneous spaces and Haar measure.

Once we have Grassmannians, it is easy to describe the set of all k planes and integration on k planes. This leads to the definition of the k -plane transform and its adjoint. We study the basic properties of the k -plane transform in sections 3.4 and 3.5.

An inversion formula for the k -plane transform is of great interest. We provide four main approaches to the inversion of the k -plane transform. Each approach results in essentially the same inversion formula; the differences are in which class of functions that can be inverted.

We begin with a very easy approach that yields an inversion formula for L^1 functions, satisfying a condition on the Fourier transform. This class includes rapidly decreasing functions and functions in $L^1 \cap L^2$ and hence most of the functions commonly used in practice in tomography. This inversion formula is developed in section 3.6.

More general inversion formulas are based on the Riesz potentials, which form a family of linear operators that we study in section 3.7. In section 3.7 we also study Fuglede’s theorem, which demonstrates a close connection between k -plane transforms

and Riesz potentials and, in fact, shows that if one could invert the Riesz potential operator, then one would have an immediate formula for the inversion of the k -plane transform.

The Riesz potential can be inverted on large classes of functions. For example, the books of Rubin [524], Samko [541], and Samko, Kilbas and Marichev [542] discuss general results on the inversion of the Riesz potential. From these results we could easily prove inversion formulas for the k -plane transform. However, the background needed would violate the prerequisites that we have established for this chapter. Therefore, in section 3.8, we develop an inversion formula for the k -plane transform based on a Fourier multiplier theorem for the Riesz potential. We are able to develop this approach using only very elementary properties of the Riesz potential along with some elementary facts about real analysis and distributions. This approach originated with the paper [564] of Smith and Keinert (for $k = 1$). The generalization to $k > 1$ was done by Keinert [348]. We have considerably simplified the proofs of these papers.

The third approach to the inversion of the k -plane transform is handled in section 3.9 by treating the k -plane transform as an unbounded operator on $L^2(\mathbb{R}^n)$.

The fourth and last approach to inversion formulas is treated in section 3.10. The inversion theorem in section 3.8 applies to a wide class of L^2 functions. In section 3.10 we describe an inversion theorem due to Rubin that applies to certain L^p functions. We describe Rubin's results without proofs because of the reasons stated earlier.

In section 3.10 we also present some properties of the k -plane transform as a bounded operator between L^p spaces.

The inversion of the k plane transform for L^p functions requires considerably more machinery than that required for L^1 functions. The reader who is only interested in inversion formulas for very well behaved functions, say compactly supported L^2 functions or rapidly decreasing functions, need only study the first inversion formula in section 3.6. However, the other inversion results are used in subsequent chapters.

As an application of the inversion formulas we derive the higher-dimensional inversion formulas that Radon stated in his 1917 paper [508]. The case of dimension $n = 2$ was already treated by other methods in chapter 2.

Roughly speaking, a Radon transform is said to be local if a function can be recovered in the neighborhood of a point by using only k planes passing close to that point. The Radon transform is local in odd dimensions but not in even dimensions. However, there are transforms related to the Radon transform that are local in even dimensions. This is studied in section 3.11 on local tomography.

Finally, in section 3.12 we investigate to what degree the k -plane transform is injective on various spaces. This leads to both uniqueness and nonuniqueness results.

The prerequisites for reading this chapter include a familiarity with the Lebesgue theory of integration and measure, the elements of group theory, the Fourier transform on $\mathbb{R}^n$, and the elementary theory of tempered distributions. Just to get to the Lebesgue theory entails having had the usual prerequisites such as advanced calculus, point-set topology, linear algebra, etc., so we will not be concerned with providing references in these areas. Ideally, the reader should also know at least the rudiments of the theory of the Haar integral and have some knowledge of Grassmann manifolds. However, we will

outline and make precise any necessary concepts from these areas and give appropriate references for the interested reader. Therefore, this chapter should be accessible to many scientists, engineers, and mathematicians who are not analysts.

Section 3.2 on notation and introductory material serves two purposes. It establishes the notational conventions used here and it serves as a reference for some of the prerequisite material. I believe that the reader should briefly go over this section to establish the notation and then try to proceed to the body of the chapter, using the introductory section only as a reference when needed.

3.2 Notation and Introductory Material

The notation follows that introduced in chapter 2. We do generalize the notation $|A|$, which in chapter 2 denoted the Lebesgue measure of a set A , to mean the volume of the set A relative to whatever measure is in current use.

As usual Euclidean n -space is denoted by $\mathbb{R}^n$ and complex Euclidean n -space is denoted by $\mathbb{C}^n$.

The unit sphere in $\mathbb{R}^n$ consists of all unit n -vectors. It is denoted by S^{n-1} and its $(n-1)$ -dimensional volume is $|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

$B^n(c, r)$ is the **open ball** of radius r centered at c in $\mathbb{R}^n$. It is defined by the inequality $|x - c| < r$. If $c = 0$ then we let $B^n(r) = B^n(0, r)$. We usually drop the superscript for the dimension if no confusion can arise. The volume of the ball is $|B^n(c, r)| = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n$.

$\text{supp}(f)$ denotes the **support** of a function f .

$\delta(K)$ denotes the **diameter** of a set K .

The **restriction of a function** f to a set S is denoted by $f|_S$.

$C^m(X)$ denotes the set of m times continuously differentiable functions on the set X . We allow m to range from 0 to ∞ . In the case $m = 0$ the convention is that $C^0(X)$ is the set of continuous functions defined on X .

If $1 \leq p < \infty$, then $L^p(X)$ denotes the **Lebesgue space** of functions f defined on the measure space X , such that $\|f\|_{L^p(X)} = [\int_X |f(x)|^p dx]^{1/p}$ is finite. We also define $L^\infty(X)$ to be the space of essentially bounded functions on X with the essential supremum norm. Therefore, if $f \in L^\infty(X)$, then $|f(x)| \leq \|f\|_{L^\infty(X)}$ almost everywhere. These norms define the structure of a Banach space on all the L^p spaces, $1 \leq p \leq \infty$. In the case $p = 2$, the space $L^2(X)$ is a Hilbert space.

We sometimes simplify the notation and write L^p for the space and $\|f\|_{L^p}$ for the norm if no confusion can arise. Particular cases are the space L^1 of integrable functions and the space L^2 of square integrable functions.

The notation $f_j \xrightarrow{L^p} f$ means that the sequence of functions f_j converges to the function f in L^p -norm. This is equivalent to $\lim_{j \rightarrow \infty} \|f_j - f\|_{L^p(X)} = 0$.

We use the notation $\langle \cdot, \cdot \rangle_F$ for an inner product with arguments in a linear space F . We use the same notation if F is a measure space and the inner product is taken in some L^p space associated to the measure space. If there is no confusion about the space F ,

then we drop the subscript F and just write $\langle \cdot, \cdot \rangle$. If the inner product is defined by integration and F includes complex valued functions, then

$$\langle f, g \rangle_F = \int f(x) \overline{g(x)} dx.$$

We have already been using the notation $\langle x, y \rangle$ for the inner product on $\mathbb{R}^n$. This is the special case where $F = \mathbb{R}^n$. The meaning of any pairing can always be gleaned from the context.

The **conjugate index** p' of a nonzero real number p is the unique real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. Let X be a measure space. **Hölder's inequality** states that if $1 \leq p \leq \infty$, $f \in L^p(X)$, and $g \in L^{p'}(X)$, then

$$|\langle f, g \rangle_X| \leq \|f\|_{L^p(X)} \|g\|_{L^{p'}(X)}. \quad (3.1)$$

This is, of course, equivalent to the integral inequality

$$\left| \int_X f(x) \overline{g(x)} dx \right| \leq \left[\int_X |f(x)|^p dx \right]^{1/p} \left[\int_X |g(x)|^{p'} dx \right]^{1/p'}.$$

One frequent application of Hölder's inequality is to show that a particular integral or inner product is finite. If the integral is expressible as a product of an L^p function times an $L^{p'}$ function, then Hölder's inequality shows that the integrand is actually Lebesgue integrable and that the integral is therefore well defined and finite.

A standard result is that $L^{p'}(X)$ is the dual or conjugate space to $L^p(X)$. Hence, any continuous linear functional λ on $L^p(X)$ has the form $\lambda(f) = \langle f, g_0 \rangle_X$, where g_0 is a uniquely defined element of $L^{p'}(X)$.

If $p = 2$, then Hölder's inequality is the same as the Cauchy–Schwarz inequality. Noting that if $p = 2$, then $p' = 2$ also, we see that in this case the $L^p - L^{p'}$ duality is just an aspect of the well known fact that a Hilbert space is naturally isomorphic to its dual space.

A **multiindex** α is an ordered n -tuple of nonnegative integers, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$.

The **differential operator** ∂^α is defined by

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

where each $\frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}$ is just the usual partial derivative operator with respect to the j -th variable of $\mathbb{R}^n$.

$\mathcal{S}$ denotes the **Schwartz class of rapidly decreasing functions** on $\mathbb{R}^n$. Functions in this class are characterized by being infinitely differentiable and by satisfying the inequalities $\sup_{x \in \mathbb{R}^n} |x^\alpha| |\partial^\beta f(x)| < \infty$ for any multiindices α, β . We will use the expressions “Schwartz class function,” “rapidly decreasing function,” and “ $f \in \mathcal{S}$ ” synonymously.

A 0 subscript on a function space F indicates the subspace of compactly supported functions in F .¹ For example, $C_0^\infty(X)$ denotes the space of compactly supported C^∞ functions on X , whereas $L_0^2(\mathbb{R}^n)$ denotes the space of compactly supported square integrable functions on $\mathbb{R}^n$.

The subscript *loc* indicates the set of functions that locally have the specified property. For example, $L_{\text{loc}}^1(\mathbb{R}^n)$ denotes the set of locally integrable functions on $\mathbb{R}^n$. This is the set of functions f defined on $\mathbb{R}^n$ such that every point of $\mathbb{R}^n$ has a neighborhood such that the restriction of f to that neighborhood is integrable.

The notation $x \mapsto f(x)$ is used as a synonym for the function f with the provision that the domain is either specified or implied to be the largest for which the expression $f(x)$ makes sense.

The **Fourier transform** which was introduced for L^1 functions in chapter 2 by

$$\widehat{f}(y) = (\mathcal{F}f)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, y \rangle} dx \quad (3.2)$$

can be extended to L^2 functions. The set of functions that are both L^1 and L^2 on $\mathbb{R}^n$ is a dense subspace of $L^2(\mathbb{R}^n)$ on which the Fourier transform is defined according to equation (3.2). It is possible to continuously extend the Fourier transform from $L^1 \cap L^2$ to $L^2(\mathbb{R}^n)$ in such a way that it becomes an isometry of L^2 . This isometry implies the **Plancherel formula**:

$$\langle f, g \rangle_{L^2} = \langle \widehat{f}, \widehat{g} \rangle_{L^2}.$$

Because of this isometry, the Fourier transform has an inverse on L^2 . This inverse is denoted by $\widetilde{f}$ or $\mathcal{F}^{-1}f$. If f is integrable we have

$$\widetilde{f}(y) = (\mathcal{F}^{-1}f)(y) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{i\langle x, y \rangle} dx.$$

We have

$$\widetilde{\widetilde{f}} = \widehat{f}$$

and from this we get a useful variant of the Plancherel formula:

$$\langle \widehat{f}, \widehat{g} \rangle_{L^2} = \langle f, \widetilde{g} \rangle_{L^2}$$

We generally use the notation $\widehat{f}$ for the Fourier transform, but if the expression defining f is long or complicated, then it is necessary to use an alternate notation: if $\cdots$ denotes a long expression, then either $\mathcal{F}(\cdots)$ or $(\cdots)^\wedge$ denotes the Fourier transform.

The convolution theorem for the Fourier transform

$$\widehat{f * g} = (2\pi)^{\frac{n}{2}} \widehat{f} \widehat{g} \quad (3.3)$$

¹ Some authors use the subscript c to denote spaces of compactly supported functions. Therefore, our C_0^∞ would be denoted as C_c^∞ .

is valid on L^1 . The function $\widehat{f}$ is called a multiplier for the operator $T(g) = f * g$ and equation (3.3) is called a **multiplier theorem**.

The **derivative theorem** for the Fourier transform states that

$$\begin{aligned}\widehat{\partial^\alpha f}(\xi) &= (i\xi)^\alpha \widehat{f}(\xi) \quad \text{and} \\ ((-ix)^\alpha f(x))^\wedge(\xi) &= (\partial^\alpha \widehat{f})(\xi).\end{aligned}\tag{3.4}$$

provided that the Fourier transforms of f and the associated derivatives exist.

The following generalization of the derivative theorem is useful. Let p be a polynomial on $\mathbb{R}^n$ and let $\partial = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ be the gradient operator. We can form the differential operator $p(\partial)$ from the polynomial $p(x_1, \dots, x_n)$ by replacing each instance of x_j by $\frac{\partial}{\partial x_j}$. Then we have the more general derivative theorem

$$\begin{aligned}(p(\partial) f)^\wedge(\xi) &= p(i\xi) \widehat{f}(\xi) \quad \text{and} \\ (p(-ix) f(x))^\wedge(\xi) &= (p(\partial) \widehat{f})(\xi).\end{aligned}\tag{3.5}$$

The **dilation theorem** for the Fourier transform states that

$$\widehat{f(ax)} = a^{-n} \widehat{f}(a^{-1}x).$$

Functions in the Schwartz class $\mathcal{S}$ of rapidly decreasing C^∞ functions are both in L^1 and in L^2 and, hence, have well-defined Fourier transforms. It is possible to show that the Fourier transform restricted to $\mathcal{S}$ is a vector space isomorphism. Also $\mathcal{S}$ is dense in any L^p space $1 \leq p < \infty$.

There is a natural topology on $\mathcal{S}$, and **tempered distributions** are defined to be continuous linear functionals on $\mathcal{S}$ (see Stein and Weiss [583]). The vector space of tempered distributions is denoted by $\mathcal{S}'$. If u is a tempered distribution, then its value $u(\varphi)$ at a test function $\varphi \in \mathcal{S}$ is a complex number, and we extend the inner product notation by defining

$$\langle u, \varphi \rangle = u(\overline{\varphi}),$$

where $\overline{\varphi}$ denotes the complex conjugate of the function φ .

A function f is said to be **slowly increasing** if f is measurable on $\mathbb{R}^n$ and if a positive integer N exists such that $\frac{f(x)}{(1+|x|^2)^N}$ is essentially bounded. Any slowly increasing function f defines a tempered distribution T_f by letting

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^n} f(x) \overline{\varphi(x)} dx.$$

In general, we define a **tempered function** to be a function f with the property that $\frac{f(x)}{(1+|x|^2)^N} \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. If g is a tempered function, then T_g is a tempered distribution. The reader can consult Stein and Weiss [583], pp. 21–22, for details.

The correspondence $f \mapsto T_f$ is one-to-one in the sense that if $T_f = T_g$, then $f = g$ almost everywhere. Therefore, we identify f with T_f . If f is a slowly increasing or

tempered function and if $\varphi \in \mathcal{S}$, then we have

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x) \overline{\varphi(x)} dx.$$

Hence, no conflict exists between the notations $\langle f, \varphi \rangle$, where f is considered as a function versus the same notation when f is considered as a distribution. The notation $\langle u, \varphi \rangle$ therefore extends the usual inner product notation to the case where u is a distribution.

On the other hand, many tempered distributions are not functions. For example, the Dirac δ distribution defined by $\langle \delta, \varphi \rangle = \varphi(0)$ is such a distribution.

The Fourier transform of a tempered distribution u is the element $\mathcal{F}(u) = \widehat{u}$ of $\mathcal{S}'$ defined by

$$\langle \widehat{u}, \varphi \rangle = \langle u, \widetilde{\varphi} \rangle. \quad (3.6)$$

This definition is motivated by the multiplication formula for Fourier transforms that states that if f and g are L^1 functions on $\mathbb{R}^n$, then (Stein and Weiss [583], p. 8) $\int_{\mathbb{R}^n} \widehat{f}(x) \overline{\widehat{g}(x)} dx = \int_{\mathbb{R}^n} f(x) \overline{g(x)} dx$, which in the notation of inner products is $\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle$.

Definition (3.2) immediately implies the *Plancherel formula* for tempered distributions:

$$\langle \widehat{u}, \widehat{\varphi} \rangle = \langle u, \varphi \rangle. \quad (3.7)$$

The Hausdorff–Young theorem is used in just a few places although this theorem may be beyond the prescribed prerequisites. The reader who is not familiar with this result can consult chapter 5 of Stein and Weiss [583].

Because L^p functions are tempered distributions, they have Fourier transforms as distributions. However, the Fourier transform of an L^p function may not itself be a function (this a consequence of theorem 7.6.6 in Hörmander [317]). On the other hand, Hausdorff and Young proved that if $1 \leq p \leq 2$, then the Fourier transform $\widehat{f}$ of any $f \in L^p(\mathbb{R}^n)$ is a function and, in fact, is in $L^{p'}(\mathbb{R}^n)$, where p' is the conjugate index to p . This may be called the **Hausdorff–Young theorem**. Furthermore, we have the **Hausdorff–Young inequality**:

$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

For convenience we refer to these results collectively as the Hausdorff–Young inequality.

Clearly, in the case $p = 2$, the Hausdorff–Young theorem and inequality are versions of the Plancherel theorem.

There is also the following version of the Plancherel theorem for functions in L^p , which we state without proof.

Theorem 3.1 (Hausdorff–Young version of the Plancherel theorem). *Let $1 \leq p \leq 2$ and $f \in L^p$. Also let $g \in L^{p'}$. Then Fourier transform of g exists as a tempered*

distribution. Suppose further that $\widehat{g}$ is not only a distribution, but that $\widehat{g}$ is a function in L^p . Then we have

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle$$

Bispherical coordinates are needed at one point. These are a generalization of spherical coordinates for the unit sphere. If n and k are natural numbers with $k < n$, then we can decompose $\mathbb{R}^n$ as follows: $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$, where $\mathbb{R}^k = \{(x_1, \dots, x_k, 0, \dots, 0)\}$ and $\mathbb{R}^{n-k} = \{(0, \dots, 0, x_{k+1}, \dots, x_n)\}$.

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then we let $x' = (x_1, \dots, x_k, 0, \dots, 0)$ and $x'' = (0, \dots, 0, x_{k+1}, \dots, x_n)$, so $x = x' + x''$ and this is a direct sum. We also note that we can consider $\mathbb{R}^n$ to be the direct sum of $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ in the sense of Hilbert spaces. In this sense, $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are orthogonal complements of each other.

We also let $S^{k-1} = S^{n-1} \cap \mathbb{R}^k$ and $S^{n-k-1} = S^{n-1} \cap \mathbb{R}^{n-k}$. A moment's reflection shows that S^{k-1} is essentially the usual unit $k-1$ sphere except that it is considered to be in the copy $\mathbb{R}^k \subset \mathbb{R}^n$ instead of the usual $\mathbb{R}^k$. The same applies to S^{n-k-1} .

Bispherical coordinates are defined by the map $\Theta : S^{k-1} \oplus S^{n-k-1} \times [0, \frac{\pi}{2}] \rightarrow S^{n-1}$ given by

$$\Theta(a, b, \psi) = \cos(\psi)a + \sin(\psi)b \quad (3.8)$$

It is easy to show that Θ actually maps onto the unit sphere and that it is injective almost everywhere. Therefore, Θ provides a new coordinate system on the unit sphere, called bispherical coordinates.

The formula for change of variables in integration under bispherical coordinates is

$$\begin{aligned} \int_{S^{n-1}} f(\theta) d\theta &= \int_{S^{k-1} \oplus S^{n-k-1} \times [0, \frac{\pi}{2}]} f(\cos(\psi)a + \sin(\psi)b) \\ &\quad \times \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) da db d\psi \end{aligned} \quad (3.9)$$

Although a proof of this formula may be found in Vilenkin and Klimyk [610], we sketch the argument by showing how to compute the pull back $\Theta^*(d\theta)$ where $d\theta$ is the volume element of S^{n-1} . However, the reader who is not familiar with differential forms and integration on manifolds should probably skip the demonstration. Flanders [176] has an introduction to differential forms and their integration.

It is known (Flanders [176]) that

$$d\theta = \sum_{j=1}^n (-1)^{j+1} x_j dx (\widehat{j}) \quad (3.10)$$

Here $dx(\widehat{j})$ denotes the $n-1$ form $dx_1 \wedge \dots \wedge dx_n$ with dx_j missing. We also let da be the volume element on the submanifold S^{k-1} , and we let db be the volume element

on the submanifold S^{n-k-1} . We then note that, on each submanifold, respectively

$$da = \sum_{j=1}^k (-1)^{j+1} a_j da(\hat{j}) \quad (3.11)$$

$$db = \sum_{j=k+1}^n (-1)^{j-k+1} b_j db(\hat{j}) \quad (3.12)$$

Letting $x = \Theta(a, b, \psi)$, from equation (3.8) it is easy to show that if $j = 1, \dots, k$, then

$$\Theta^*(dx_j) = -\sin(\psi) a_j d\psi + \cos(\psi) da_j$$

and if $j = k+1, \dots, n$, then

$$\Theta^*(dx_j) = \cos(\psi) b_j d\psi + \sin(\psi) db_j.$$

Applying the pullback Θ^* to equation (3.10) yields

$$\begin{aligned} \Theta^*(d\theta) &= \sum_{j=1}^k (-1)^{j+1} \cos(\psi) a_j (\Theta^*(dx))(\hat{j}) \\ &\quad + \sum_{j=k+1}^n (-1)^{j+1} \sin(\psi) b_j (\Theta^*(dx))(\hat{j}). \end{aligned} \quad (3.13)$$

An elementary, but tedious, calculation shows that

$$\Theta^*(dx)(\hat{j}) = \cos(\psi) \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) da(\hat{j}) \wedge d\psi \wedge db$$

if $j = 1, \dots, k$, and

$$\Theta^*(dx)(\hat{j}) = -\sin(\psi) \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) d\psi \wedge da \wedge db(\hat{j})$$

if $j = k+1, \dots, n$. Substituting these expressions in the equation for $\Theta^*(d\theta)$ yields

$$\begin{aligned} \Theta^*(d\theta) &= \cos^2(\psi) \left(\sin^{n-k-1}(\psi) \cos^{k-1}(\psi) \right) \\ &\quad \times \left(\sum_{j=1}^k (-1)^{j+1} a_j da(\hat{j}) \right) \wedge d\psi \wedge db \\ &\quad - \sin^2(\psi) \left(\sin^{n-k-1}(\psi) \cos^{k-1}(\psi) \right) \\ &\quad \times d\psi \wedge da \wedge \left(\sum_{j=k+1}^n (-1)^{j+1} b_j db(\hat{j}) \right) \end{aligned}$$

Equations (3.11) and (3.12) allow us to replace the summations by da and $(-1)^k db$, respectively. This yields

$$\begin{aligned} \Theta^*(d\theta) &= \cos^2(\psi) \left(\sin^{n-k-1}(\psi) \cos^{k-1}(\psi) \right) da \wedge d\psi \wedge db \\ &\quad - (-1)^k \sin^2(\psi) \left(\sin^{n-k-1}(\psi) \cos^{k-1}(\psi) \right) d\psi \wedge da \wedge db. \end{aligned}$$

Commuting the exterior product $da \wedge d\psi$ introduces a factor of $(-1)^{k-1}$ and the pull-back then simplifies to

$$\Theta^*(d\theta) = (-1)^{k-1} \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) d\psi \wedge da \wedge db \quad (3.14)$$

The theorem for changing coordinates in an integral over a manifold takes the following form (Flanders [176]):

$$\int_{\gamma} \Theta^* \omega = \int_{\Theta_* \gamma} \omega$$

If we suitably orient the manifolds in question, then the factor $(-1)^{k-1}$ disappears. We can take $\gamma = S^{k-1} \oplus S^{n-k-1} \times [0, \frac{\pi}{2}]$, in which case $\Theta_* \gamma = S^{n-1}$ and we then get formula (3.9) for change of variable by bispherical coordinates.

3.2.1 References for the Introductory Material

The reader may consult the following references concerning the concepts introduced here. Hewitt and Stromberg [304] is a comprehensive text containing most of the background material required on the Lebesgue theory and Fourier transforms, but not on distributions. For readers already familiar with the basic Lebesgue theory, Stein and Weiss [583], chapter 1, has an excellent and concise introduction to the Fourier transform on L^p spaces, to the Schwartz class and to tempered distributions. Helgason [291] (chapter 5, in the second edition, 1999) develops tempered distributions, the Fourier transform for functions in the Schwartz class, and the Fourier transform for tempered distributions. For the reader who is not familiar with meromorphic extensions of distributions, the discussion in sections 2–4 of Jensen [321] may be helpful in conjunction with reading chapter 5 of Helgason [291]. This discussion parallels Helgason’s but is somewhat more elementary and has more details.

3.3 Grassmann Manifolds, Homogeneous Spaces, and Haar Measure

Any translation of a k -dimensional vector subspace of $\mathbb{R}^n$ is called a k **plane**. An $n - 1$ plane is therefore exactly the same as a hyperplane (compare, chapter 2) and a 1 plane is a straight line. The generalization of the Radon transform treated in this chapter introduces a family of transforms, one for each $k = 1, \dots, n - 1$. The k th of these transforms integrates functions over k planes, in the same way that the Radon transform integrates functions over hyperplanes. This is called the k -**plane transform**, and the Radon transform is essentially the $n - 1$ -plane transform. We also call the k -plane transform the k -**dimensional Radon transform** or the **Radon–John transform**.

The set of all k -dimensional vector subspaces of $\mathbb{R}^n$ will play a fundamental role here. For convenience we introduce the term k -**space** as a synonym for “ k -dimensional vector subspace of $\mathbb{R}^n$.” The set of k -spaces is denoted by $G_{k,n}$ and is called a **Grassmann manifold** or a **Grassmannian**. It can be endowed both with the structure of a

differentiable manifold and the structure of a measure space. The differentiable structure is not important for us, but the ability to integrate over $G_{k,n}$ is essential.

For a given Grassmannian $G_{k,n}$ we let η represent a variable in this space. Therefore, η is a k -space and $\eta^\perp$ denotes its orthogonal complement. Hence, $\eta^\perp \in G_{n-k,n}$. We will shortly introduce the idea of Haar measure and show that Grassmannians have a Haar measure. If η is a variable in $G_{k,n}$ then we use $d\eta$ to represent the Haar measure of $G_{k,n}$. We also establish the convention that variables of the form x' live in η but those of the form x'' live in $\eta^\perp$. Another way of thinking of this is to let x be a variable in $\mathbb{R}^n$ and to let η be a k -space in $G_{k,n}$. Then x' is the orthogonal projection of x onto η and x'' is the orthogonal projection of x onto $\eta^\perp$.

Remark 3.2. A good reference for more information on Grassmannians is Santaló [543].

Remark 3.3. A natural identification exists between $G_{k,n}$ and $G_{n-k,n}$ that is based on the natural duality between k -spaces and their $n - k$ -dimensional orthogonal complements.

Remark 3.4. $G_{0,n}$ is the manifold of all 0-dimensional linear subspaces of $\mathbb{R}^n$. There is only one such subspace, so $G_{0,n}$ consists only of a single point as does $G_{n,n}$ by the comment on duality. $G_{1,n}$ is the manifold of all lines through the origin in $\mathbb{R}^n$. This is, of course, real projective space $\mathbb{RP}^{n-1}$. We can therefore think of $G_{1,n}$ as the unit sphere S^{n-1} with pairs of diametrically opposed points identified. By duality, we can identify $G_{n-1,n}$ with $\mathbb{RP}^{n-1}$. This is commonly done by many authors who study the Radon transform.

We denote the identity element of a group by e , except in the case of matrix groups where we use I to denote the identity.

A **topological group** is both a group and a topological space for which the group operations are continuous.

If G is a group and H is a subgroup, then the left **cosets** consist of the left multiples of H by fixed elements of G . A typical left coset is denoted by gH . The set of all left cosets of H is denoted by the symbol G/H and is called the quotient set of G by H . We also call G/H a **homogeneous G -set**, or simply a **homogeneous set**. Recall that the quotient set inherits a group structure from G if and only if H is a normal subgroup of G . However, we are not restricting ourselves to normal subgroups, so we merely consider G/H as a set. However, if G is also a topological group, then we make G/H into a topological space by giving it the quotient topology. This is the topology with the most open sets that makes the mapping $g \mapsto gH$ continuous.

If G is a topological group, then the quotient G/H , endowed with the quotient topology is called a **homogeneous space**.

We can envision homogeneous spaces differently. Suppose that there is a continuous map $G \times X \rightarrow X$, where G is a topological group and where X is a topological space. We let gx denote the value of this map at (g, x) for $g \in G$ and $x \in X$. We say that this

map is a **group action** on X or that G **acts on** X , if for all $g, h \in G$ and $x \in X$ we have

$$\begin{aligned} g(hx) &= (gh)x \\ ex &= x. \end{aligned}$$

The group action is said to be **transitive** if there is some $x_0 \in X$, such that for every $y \in X$ there exists some $g \in G$ such that $gx_0 = y$. It is easy to show that if the group action is transitive then any element of X can play the role of the element x_0 in this definition.

We can now show that topological spaces with transitive group actions are essentially the same as homogeneous spaces. Let X be a topological space and let G be a topological group that acts transitively on X . Let x_0 be any fixed element of X . Consider the subset H of G defined by

$$H = \{h \in G : hx_0 = x_0\}.$$

This is a subgroup of G because it contains the identity and because, if $h, j \in H$, then $(h^{-1}j)x_0 = h^{-1}(jx_0) = h^{-1}x_0 = x_0$. We call H the **isotropy group** of x_0 and we call x_0 the **center** of the isotropy. We can now define a mapping $\Gamma : G/H \rightarrow X$ by $\Gamma(gH) = gx_0$. This mapping is well defined since $g_1H = g_2H$, if and only if $g_2^{-1}g_1 \in H$, which then implies $g_1x_0 = g_2((g_2^{-1}g_1)x_0)$. But $g_2^{-1}g_1$ is in the isotropy group so we immediately obtain $g_1x_0 = g_2x_0$. This shows that the mapping Γ is well defined. A very similar proof shows that it is one-to-one. The assumption that the group action is transitive immediately shows that the mapping Γ is surjective, whereas the assumption that the group action is continuous shows that the map Γ is also continuous. We thus obtain a continuous, one-to-one, and surjective mapping of the homogeneous space G/H onto the topological space X . If we further assume that G is compact, a condition that will be valid in all our applications of homogeneous sets, then we see that we actually have a homeomorphism between G/H and X . Therefore we will not distinguish between quotient sets G/H and spaces X with a transitive group action, at least in the compact case; we refer to them, in either incarnation, as homogeneous spaces.

We next show that a sphere can be represented as a homogeneous space.

3.3.1 The Unit Sphere as a Homogeneous Space

The set $GL(n)$ of all nonsingular $n \times n$ matrices is a group under the operation of matrix multiplication. It is called the **general linear group** and is in one-to-one correspondence with a subset Y of $\mathbb{R}^{n^2}$; for any such matrix A associate the vector in $\mathbb{R}^{n^2}$ formed by concatenating the columns of A . We can then take the metric space structure induced on Y by $\mathbb{R}^{n^2}$ and transfer it to $GL(n)$. In particular, the Heine–Borel theorem is valid; the compact subsets of $GL(n)$ are precisely the closed, bounded subsets. Under this topology $GL(n)$ is a topological group.

In dealing with matrices we establish the convention that if A is a matrix, then A_{ij} denotes the entry in the i th row and j th column of A and A_j denotes the j th column of A .

An **orthogonal matrix** is an $n \times n$ matrix whose columns form an orthonormal basis of $\mathbb{R}^n$. Let $O(n)$ denote the set of orthogonal $n \times n$ matrices. In a moment we show that with matrix multiplication as the product, this set forms a group that we call the **orthogonal group**.

In the case $n = 1$ the orthogonal group consists of $\{-1, 1\}$ and in the case $n = 0$ we make the special definition $O(0) = \{1\}$.

It is easy to show that an $n \times n$ matrix A is orthogonal if and only if $A^T = A^{-1}$. From this relation it immediately follows that orthogonal matrices are closed under matrix multiplication and, hence, $O(n)$ inherits the structure of a topological group from $GL(n)$. By the continuity of the group operations, the function $A \mapsto A^T - A^{-1}$ is continuous on $GL(n)$. Therefore, its zero set $O(n)$ is closed. The fact that the columns of orthogonal matrices are unit vectors shows that $O(n)$ is bounded in $GL(n)$. Hence, $O(n)$ is a compact topological group.

Subgroups of the orthogonal group generate many interesting homogeneous spaces. For example, the unit sphere S^{n-1} of $\mathbb{R}^n$ can be identified with the homogeneous space $O(n)/O(n-1)$, if $n \geq 1$. To see this we define the group action that sends $(A, \theta) \in O(n) \times S^{n-1}$ to $A\theta$. That this is a valid group action and that it is transitive is easily checked. Therefore, S^{n-1} is topologically the same as $O(n)/H$, where H is the isotropy group of the action.

To identify the isotropy group, let us use $e_1 = (1, 0, 0, \dots, 0)$ as the center. The isotropy group then consists of all orthogonal matrices A with $Ae_1 = e_1$. But this is precisely the set of all orthogonal matrices whose first column is e_1 . Because A is orthogonal we know that if A_j is a column vector of A with $j > 1$, then $A_{1j} = \langle A_j, e_1 \rangle = 0$. This means that the first row of A contains all zeros except for the A_{11} entry. Because the first column is e_1 , the A_{11} entry must be 1. Hence, letting $O_{j \times k}$ denote the $j \times k$ zero matrix, there exists a $1 \times (n-1)$ matrix A'' and an $(n-1) \times (n-1)$ matrix A' , such that A has the form $A = \begin{bmatrix} 1 & A'' \\ O_{(n-1) \times 1} & A' \end{bmatrix}$. Since the first row of A is a unit vector with first entry 1, then we must have $A'' = O_{1 \times (n-1)}$. Hence, $A = \begin{bmatrix} 1 & O_{1 \times (n-1)} \\ O_{(n-1) \times 1} & A' \end{bmatrix}$. The structure of this matrix together with its orthogonality shows that A' is an $(n-1) \times (n-1)$ orthogonal matrix. Therefore, the isotropy group has the form

$$\left\{ \begin{bmatrix} 1 & O_{1 \times (n-1)} \\ O_{(n-1) \times 1} & A' \end{bmatrix} : A' \in O(n-1) \right\} \quad (3.15)$$

where A' is an $(n-1) \times (n-1)$ orthogonal matrix. From equation (3.15) it follows that the isotropy group is isomorphic to $O(n-1)$, and we make this identification from now on.

From our results on homogeneous spaces and the fact that $O(n)$ is compact, we have proved the following result for $n > 1$. The case $n = 1$ is easy to establish directly from the definitions (compare, Chevalley [90]).

Theorem 3.5. *If n is a natural number, then there is a homeomorphism between $O(n)/O(n-1)$ and S^{n-1} .*

Each element θ of the unit sphere S^{n-1} determines an $n - 1$ -dimensional vector subspace of $\mathbb{R}^n$. Therefore, it is not surprising that a generalization of the procedure for the sphere will show that any Grassmannian is a homogeneous space.

3.3.2 Grassmann Manifolds as Homogeneous Spaces

We now give the Grassmannian $G_{k,n}$ the structure of a homogeneous space. Although we have not placed a topological structure on Grassmannians, we will show that there is a one-to-one surjective map of the homogeneous space $O(n)/O(k) \times O(n-k)$ onto $G_{k,n}$. We then carry over the quotient topology from $O(n)/O(k) \times O(n-k)$ to $G_{k,n}$, making the Grassmannian a compact homogeneous space.

We begin by defining the action of the orthogonal group on the Grassmannian by assigning to $(A, \eta) \in O(n) \times G_{k,n}$ the k -space $A\eta$. The symbol $A\eta$ denotes the image of the k -dimensional vector space η under the linear transformation defined by the matrix A . Since $A \in O(n)$, we see that $A\eta$ is also k -dimensional, so the mapping just described goes from $O(n) \times G_{k,n}$ to $G_{k,n}$. This together with the easily checked fact that $A(B\eta) = (AB)\eta$ and $I\eta = \eta$ for $A, B \in O(n)$ and $\eta \in G_{k,n}$ shows that this mapping satisfies the definition of a group action.

To show that this action is transitive, we single out the k -space η_0 , which is the span of the first k -standard unit vectors $e_1, \dots, e_k$ of $\mathbb{R}^n$. If we are now given an arbitrary k -space η of $\mathbb{R}^n$, then we can use the Gram–Schmidt process to find an orthonormal basis $\{A_1, \dots, A_k\}$ of η and an orthonormal basis $\{A_{k+1}, \dots, A_n\}$ of the orthogonal complement $\eta^\perp$ of η . Define the matrix A to be the one whose columns are the vectors A_j , $j = 1, \dots, n$, which we just chose. By the way we constructed these vectors, A is an orthogonal matrix. Finally, since $Ae_j = A_j$ we see that $A\eta_0 = \eta$. This proves that the action is transitive. Therefore, we have a one-to-one surjective map from $O(n)/H$ onto $G_{k,n}$, where H is the isotropy group of the action with center η_0 .

We now determine the structure of the isotropy group. Using the notation $O_{j \times k}$ introduced previously, we can show that

$$H = \left\{ \begin{bmatrix} B'' & O_{k \times (n-k)} \\ O_{(n-k) \times k} & B' \end{bmatrix} \in O(n) : B'' \in O(k), B' \in O(n-k) \right\} \quad (3.16)$$

To do this, let $A \in O(n)$ be such that $A\eta_0 = \eta_0$. From this and the definition of η_0 we see that for each $j = 1, \dots, k$, the j th column A_j of A is a linear combination of the standard basis vectors $e_1, \dots, e_k$. This means that the last $n - k$ entries of each such column will consist of zeros. Therefore there is a $k \times k$ matrix B'' such that the first k columns of A have the form $\begin{bmatrix} B'' \\ O_{(n-k) \times k} \end{bmatrix}$. Taking into account that A is orthogonal, this implies that $B'' \in O(k)$.

Now let us investigate the ij entry of A for the first k rows and the last $n - k$ columns. Such an entry can be expressed as $A_{ij} = \langle Ae_j, e_i \rangle = \langle e_j, A^T e_i \rangle$, where $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. Because A is orthogonal we have $A^T = A^{-1}$ so the relation becomes $A_{ij} = \langle e_j, A^{-1} e_i \rangle$. But the isotropy condition $A\eta_0 = \eta_0$ implies $A^{-1}\eta_0 = \eta_0$ so $A^{-1}e_i$ is a linear combination of the vectors $e_1, \dots, e_k$. Because $j > k$ we get $\langle e_j, e_i \rangle = 0$

for $t = 1, \dots, k$ and, hence, $A_{ij} = \langle e_j, A^{-1}e_i \rangle = 0$ for $1 \leq i \leq k$ and $k+1 \leq j \leq n$. This proves that the last $n-k$ columns of A have the structure $\begin{bmatrix} O_{k \times (n-k)} \\ B' \end{bmatrix}$, where $B' \in O(n-k)$. Hence, the isotropy group is contained in the group that is the right-hand side of (3.16). It is easy to check that any matrix in the right-hand side of (3.16) has the property $A\eta_0 = \eta_0$. This establishes the equation (3.16).

From the structure of H we see that there is a natural isomorphism $O(k) \times O(n-k) \approx H$ defined by sending $(B'', B') \in O(k) \times O(n-k)$ to the matrix $\begin{bmatrix} B'' & O_{k \times (n-k)} \\ O_{(n-k) \times k} & B' \end{bmatrix}$. This establishes a one-to-one correspondence between the homogeneous space $O(n)/O(k) \times O(n-k)$ and the Grassmannian $G_{k,n}$.

As mentioned previously, we allow $G_{k,n}$ to inherit the topological structure of $O(n)/O(k) \times O(n-k)$. With only a slight abuse of language we have the following result.

Theorem 3.6. *The Grassmannian $G_{k,n}$ can be represented as the homogeneous space*

$$G_{k,n} = O(n)/O(k) \times O(n-k). \quad (3.17)$$

In particular, $G_{1,n}$ and $G_{n-1,n}$ can be identified with $O(n)/\mathbb{Z}_2 \times O(n-1)$.

Proof. Everything has been proved, except the last statement, which follows from the fact that $O(1)$, the group of orthogonal 1×1 matrices, consists of the multiplicative group $\{-1, 1\}$ which is isomorphic to $\mathbb{Z}_2$. ■

Remark 3.7. The following assertions depend on some facts about Lie groups and their homogeneous spaces. References for these facts can be found in Chevalley [90] and Auslander and MacKenzie [28].

1. Lie groups are topological groups that are differentiable manifolds and for which the group actions are C^∞ functions.
2. Homogeneous spaces of Lie groups inherit the structure of a differentiable manifold from the group.
3. If H is a connected closed Lie subgroup of the Lie group G , then

$$\dim G/H = \dim G - \dim H.$$

4. The orthogonal group $O(n)$ is a Lie group of dimension $\frac{1}{2}n(n-1)$.

From these considerations we see that the Grassmannians $G_{k,n}$ are differentiable manifolds of dimension $k(n-k)$.

Although these are interesting results, we have no further need of them.

3.3.3 Haar Measure

We now outline some results from the theory of Haar measure. A classic and beautiful reference for the Haar integral and Lebesgue integration is Nachbin [433].

To avoid a digression on unimodular groups, we state all results about Haar measure for the case of compact homogeneous spaces.

A left invariant **Haar measure** is a measure ν , which is defined on a compact homogeneous space X and is left invariant with respect to the group action. This means that

$$\int_{x \in X} f(gx) d\nu(x) = \int_{x \in X} f(x) d\nu(x)$$

for all elements g of the group and all integrable functions f with respect to the measure ν .

We only require the following two results from the theory of Haar measure.

1. Every compact homogeneous space has a Haar measure.
2. A Haar measure is unique up to a multiplicative constant.

The uniqueness is meant in the sense that if μ and ν are two Haar measures on the same homogeneous space X , then there exists a constant c such that $\mu = c\nu$.

The general theory of Haar measures distinguishes between left and right invariant Haar measures and this may be an important distinction if the group is locally compact and not compact and not Abelian. However, we only require left invariant Haar measures and we simply refer to these as Haar measures.

3.3.4 Haar Measure on Grassmannians

We already established that every orthogonal group $O(m)$ is compact. Since $G_{k,n} = O(n)/O(k) \times O(n-k)$, then all Grassmannians are compact homogeneous spaces. As such every Grassmannian has an $O(n)$ invariant Haar measure which we denote by $\nu_{k,n}$. When we integrate over a Grassmannian the differential in the integral is denoted by $d\nu_{k,n}$, or, more informally, by $d\eta$ where η has been specified as a variable point in $G_{k,n}$.

The measure $\nu_{k,n}$ is unique up to a constant and this constant is completely determined by the measure of the entire space $|G_{k,n}|$. No universal agreement exists on the choice of the total volume $|G_{k,n}|$ of a Grassmannian. We adopt the following convention:

$$|G_{k,n}| = \frac{|S^k|}{|S^n|} \prod_{j=0}^k \frac{|S^{n-j}|}{|S^j|} \quad (3.18)$$

where $|S^m| = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})}$ is the standard Lebesgue measure of the unit m sphere. It is easy to check that the following recursion relations are true:

$$|G_{k,n}| = \frac{|S^{n-k}|}{|S^{k-1}|} |G_{k-1,n}| \quad (3.19)$$

$$|G_{k,n}| = \frac{|S^{n-1}|}{|S^{k-1}|} |G_{k-1,n-1}| \quad (3.20)$$

and that the following special cases hold:

$$\begin{aligned}
 |G_{0,n}| &= 1 \\
 |G_{1,n}| &= \frac{1}{2} |S^{n-1}| \\
 |G_{k,n}| &= \frac{1}{2} |S^{n-1}| \prod_{j=1}^{k-1} \frac{|S^{n-j-1}|}{|S^j|} \\
 &= \frac{1}{2} \frac{|S^{n-1}| |S^{n-2}| \cdots |S^{n-k}|}{|S^{k-1}| \cdots |S^1|} \quad \text{for } k \geq 2.
 \end{aligned}$$

One can also easily check that

$$|G_{k,n}| = |G_{n-k,n}|, \quad (3.21)$$

which reflects the natural duality between k -spaces and their $n - k$ -dimensional orthogonal complements. We previously showed that $G_{0,n}$ consisted of a single point. The 0-dimensional volume of a single point is 1, which is consistent with our definition of $|G_{0,n}|$. Also every line through the origin, a typical element of $G_{1,n}$, corresponds to exactly two unit vectors, which are elements of S^{n-1} . Therefore, choosing the measure of $|G_{1,n}|$ to be half the measure of the unit $n - 1$ sphere is reasonable, although some authors define $|G_{1,n}| = |S^{n-1}|$ and also omit the factor $\frac{1}{2}$ for higher-order Grassmannians. Another reason for our choice is that the unit $n - 1$ sphere is $O(n)/O(n - 1)$, whereas the Grassmannian $G_{1,n}$ is $O(n)/\mathbb{Z}_2 \times O(n - 1)$. Because $\mathbb{Z}_2 = \{-1, 1\}$ we see that, intuitively speaking, $G_{1,n}$ is half of S^{n-1} .

As our first application of Haar measure we establish a useful mapping property of Grassmannians in proposition 3.8. The next few paragraphs introduce some ideas necessary for the proof.

We show that there is a one-to-one measure preserving correspondence between the Grassmannians $G_{k,n}$ and $G_{n-k,n}$. The idea behind this result is very simple: we define the correspondence by assigning to every k -space η its orthogonal complement $\eta^\perp$. However, why should this be a measure-preserving operation?

To understand this we view Grassmannians as homogeneous spaces of the matrix group $O(n)$ as described earlier. The correspondence assigning $\eta^\perp$ to η can be moved up to $O(n)$ by assigning to an orthogonal matrix A the matrix B obtained by doing a circular shift to the right by $n - k$ columns. This shifting operation just interchanges the positions of orthonormal bases for a k -space η its orthogonal complement. Recalling the identification $G_{k,n} = O(n)/O(k) \times O(n - k)$, we see that this assignment, $A \longleftrightarrow B$, is the desired one-to-one correspondence once we pass to the coset spaces $G_{k,n} = O(n)/O(k) \times O(n - k)$ and $G_{n-k,n} = O(n)/O(n - k) \times O(k)$, since the vector space corresponding to matrix B is the one spanned by the first $(n - k)$ columns of B . Hence, the assignment $A \longleftrightarrow B$ in $O(n)$ corresponds to the assignment $\eta \longleftrightarrow \eta^\perp$ from $G_{k,n}$ to $G_{n-k,n}$.

The problem is that we must show that this correspondence is a homeomorphism that preserves the measure of corresponding subsets of the two Grassmannians. To do

this we systematize the process of moving columns and rows of matrices via circular shifts. This can be implemented by certain matrix multiplications.

We also need to use the Riesz representation theorem, which requires looking at linear functionals on $C^0(X)$, the space of continuous functions on the topological space X . Hewitt and Stromberg [304] have a statement and proof of the Riesz representation theorem.

Proposition 3.8. *The map $\Phi(\eta) = \eta^\perp$ is a one-to-one measure preserving homeomorphism between the Grassmannians $G_{k,n}$ and $G_{n-k,n}$.*

This means that if f is any integrable function on $G_{n-k,n}$, then $f \circ \Phi$ is an integrable function on $G_{k,n}$ and we have

$$\int_{G_{k,n}} f(\eta^\perp) dv_{k,n}(\eta) = \int_{G_{n-k,n}} f(\xi) dv_{n-k,n}(\xi).$$

Proof. Let S be the $n \times n$ matrix obtained from the identity matrix by performing a right circular shift of its columns, so that the last column on the right becomes the first column on the left and all other columns move to the right. It is obvious, therefore, that $S \in O(n)$. If we do a circular shift of the rows of I such that the first row becomes the last row and the other rows move up by one, then it is easy to see that we arrive at exactly the same matrix S in which the shift was by columns. We call the rowwise shift an upward circular shift.

We leave to the reader the task of checking that AS performs a right circular shift of the columns of A and that SA performs an upward circular shift of the rows of A (this follows easily from the fact that in a matrix product AB , the matrix A performs row operations on B , whereas the matrix B performs column operations on A). Right and upward circular shifts through m columns or rows are accomplished by the matrix products AS^m and $S^m A$, respectively.

Consider the map $\varphi : O(n) \rightarrow O(n)$ defined by

$$A \mapsto S^k A S^{n-k}.$$

This map is well defined because all the matrices involved are orthogonal and it is also one-to-one, surjective, and continuous. Therefore, since $O(n)$ is compact, φ is also a homeomorphism.

Note that

$$S^{n-k} S^k = I \tag{3.22}$$

because the left-hand side can be thought of as first doing a right circular shift of the identity matrix through k columns followed by a similar shift through $n - k$ columns. Because $k + (n - k) = n$, this brings us straight back to the identity matrix and establishes equation (3.22). This fact immediately implies that φ is a homomorphism, since

$$\begin{aligned} \varphi(AB) &= S^k (AB) S^{n-k} = (S^k A S^{n-k}) (S^k B S^{n-k}) \\ &= \varphi(A) \varphi(B) \end{aligned}$$

Because we already know that φ is injective and surjective, it follows that φ is an automorphism of the orthogonal group.

So far we have shown that φ is a homeomorphic automorphism of the orthogonal group. We next show that its restriction to $O(k) \times O(n-k)$ is a group isomorphism from $O(k) \times O(n-k)$ to its image $O(n-k) \times O(k)$ under φ . To do this we compute the action of φ on a matrix $A = \begin{bmatrix} B' & O'' \\ O' & B'' \end{bmatrix} \in O(k) \times O(n-k)$ (here, $B' \in O(k)$ and $B'' \in O(n-k)$; compare, equation (3.16)). The image of φ on A is $S^k A S^{n-k}$. Now $A S^{n-k}$ performs a right circular shift of length $n-k$ on the columns of A so we get

$$A S^{n-k} = \begin{bmatrix} O'' & B' \\ B'' & O' \end{bmatrix}$$

and then multiplying this on the left by S^k performs an upward circular shift of length k on the rows so we end up with

$$\varphi(A) = \begin{bmatrix} B'' & O' \\ O'' & B' \end{bmatrix} \in O(n-k) \times O(k).$$

We have just showed that $\varphi(O(k) \times O(n-k)) \subset O(n-k) \times O(k)$. Exactly the same argument interchanging the roles of k and $n-k$ proves the reverse inclusion, so we have obtained

$$\varphi(O(k) \times O(n-k)) = O(n-k) \times O(k).$$

Therefore the isomorphism φ induces a homeomorphism $\Phi : O(n)/O(k) \times O(n-k) \rightarrow O(n)/O(n-k) \times O(k)$ by defining

$$\Phi([A]) = [\varphi(A)],$$

where $[B]$ denotes the coset of B , with respect to the appropriate subgroup. From the discussion just before the statement of this proposition, we see that $\Phi(\eta) = \eta^\perp$. Also, because of our representation of the Grassmannians as quotient spaces, equation (3.17), we see that $\Phi : G_{k,n} \rightarrow G_{n-k,n}$ is a homeomorphism.

Now define a functional λ on $C^0(G_{n-k,n})$ as follows: let f be a continuous function defined on $G_{n-k,n}$ and define

$$\lambda(f) = \int_{G_{k,n}} f(\Phi(\eta)) dv_{k,n}(\eta).$$

Then λ is clearly a positive linear functional on $C^0(G_{n-k,n})$. By the Riesz representation theorem there is a unique measure μ on $G_{n-k,n}$ such that

$$\lambda(f) = \int_{G_{n-k,n}} f(\xi) d\mu(\xi)$$

for all continuous functions f on $G_{n-k,n}$. Therefore

$$\int_{G_{n-k,n}} f(\xi) d\mu(\xi) = \int_{G_{k,n}} f(\Phi(\eta)) dv_{k,n}(\eta). \quad (3.23)$$

We now check that μ is a left invariant measure on $G_{n-k,n}$. To do this choose any $A \in O(n)$. Then we need to compute $\int_{G_{n-k,n}} f(A\xi) d\mu(\xi)$. Define a function g on $G_{n-k,n}$ by

$$g(\xi) = f(A\xi).$$

It is then immediate that $f(\xi) = g(A^{-1}\xi)$.

We now prove:

$$\int_{G_{k,n}} f(\Phi(\eta)) dv_{k,n}(\eta) = c \int_{G_{n-k,n}} f(\xi) dv_{n-k,n}(\xi). \quad (3.24)$$

Using (3.23) applied to g :

$$\begin{aligned} & \int_{G_{n-k,n}} f(A\xi) d\mu(\xi) \\ &= \int_{G_{n-k,n}} g(\xi) d\mu(\xi) = \int_{G_{k,n}} g(\Phi(\eta)) dv_{k,n}(\eta) \\ &= \int_{G_{k,n}} f(A\Phi(\eta)) dv_{k,n}(\eta) = \int_{G_{k,n}} f(\Phi(\eta)) dv_{k,n}(\eta). \end{aligned} \quad (3.25)$$

The last step is by the invariance of the Haar measure $v_{k,n}$. We note that the right-hand sides of equations (3.23) and (3.25) are identical. Therefore, the left-hand sides are the same and we have $\int_{G_{n-k,n}} f(A\xi) d\mu(\xi) = \int_{G_{n-k,n}} f(\xi) d\mu(\xi)$. This proves the left invariance of the measure μ under the $O(n)$ group action.

Now that we know that both μ and $v_{n-k,n}$ are left invariant measures on the same homogeneous space, $G_{n-k,n}$, the theory of Haar measure implies that there must be a constant c such that

$$\int_{G_{n-k,n}} f(\xi) d\mu(\xi) = c \int_{G_{n-k,n}} f(\xi) dv_{n-k,n}(\xi). \quad (3.26)$$

If we combine equations (3.25) and (3.26) we then obtain equation (3.24).

If we take $f = 1$ on $G_{n-k,n}$ and use the facts that $\int_{G_{k,n}} dv_{k,n}(\eta) = |G_{k,n}|$ and $\int_{G_{n-k,n}} dv_{n-k,n}(\xi) = |G_{n-k,n}|$ in equation (3.24), then it is immediate that $c = 1$ and that the theorem is therefore true. ■

A consequence of proposition 3.8 is the following very useful generalization of the formula for changing to polar coordinates in an integral. This result is due to Solmon [569].

Theorem 3.9 (Generalized polar coordinates). *If f is a nonnegative measurable function on $\mathbb{R}^n$, or if f is integrable on $\mathbb{R}^n$, then*

$$\begin{aligned} \int_{G_{k,n}} \int_{\eta} |x'|^{n-k} f(x') dx' d\eta &= |G_{k-1,n-1}| \int_{\mathbb{R}^n} f(x) dx \\ \int_{G_{k,n}} \int_{\eta^\perp} |x''|^k f(x'') dx'' d\eta &= |G_{k,n-1}| \int_{\mathbb{R}^n} f(x) dx \end{aligned}$$

Furthermore,

$$\int_{S^{n-1}} f(\theta) d\theta = \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} \int_{S^{n-1} \cap \eta} f(\theta) d\theta d\eta. \quad (3.27)$$

Proof. First we prove that if f is a nonnegative, measurable function on S^{n-1} or if f is integrable there, then equation (3.27) is valid.

Consider the positive linear functional λ defined on $C^0(S^{n-1})$ by the right-hand side of equation (3.27). Letting A be an orthogonal matrix and $f_A(\theta) = f(A\theta)$, it follows that

$$\lambda(f_A) = \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} \int_{S^{n-1} \cap \eta} f(A\theta) d\theta d\eta.$$

Consider the change of variable $\tau = A\theta$ in the inner integral. The Jacobian of this coordinate change is 1, because A is an orthogonal matrix. Hence, we get

$$\lambda(f_A) = \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} \int_{S^{n-1} \cap A\eta} f(\tau) d\tau d\eta.$$

Define the function g by the formula $g(\eta) = \int_{S^{n-1} \cap \eta} f(\tau) d\tau$. Therefore, we can write the previous step as

$$\begin{aligned} \lambda(f_A) &= \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} g(A\eta) d\eta = \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} g(\eta) d\eta \\ &= \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} \int_{S^{n-1} \cap \eta} f(\tau) d\tau d\eta = \lambda(f) \end{aligned}$$

using the left invariance of the Haar measure $d\eta$ for the Grassmannian $G_{k,n}$ and the definition of g and λ . This shows that the measure μ associated with the functional λ by the Riesz representation theorem is invariant with respect to the group action on S^{n-1} . This means that μ is a Haar measure for the sphere and hence, there is a constant c such that $\lambda(f) = c \int_{S^{n-1}} f(\theta) d\theta$.

Now set $f \equiv 1$ on the sphere. On the one hand,

$$\lambda(1) = c \int_{S^{n-1}} d\theta = c |S^{n-1}|$$

and on the other hand

$$\lambda(1) = \frac{1}{|G_{k-1,n-1}|} \int_{G_{k,n}} \int_{S^{n-1} \cap \eta} d\theta \, d\eta.$$

But $\int_{S^{n-1} \cap \eta} d\theta = |S^{k-1}|$, because $S^{n-1} \cap \eta$ is isometric to the S^{k-1} sphere, so

$$\int_{G_{k,n}} \int_{S^{n-1} \cap \eta} d\theta \, d\eta = |S^{k-1}| \int_{G_{k,n}} \int_{S^{n-1} \cap \eta} d\eta = |S^{k-1}| |G_{k,n}|.$$

Combining these equations gives

$$c = \frac{|S^{k-1}|}{|S^{n-1}|} \frac{|G_{k,n}|}{|G_{k-1,n-1}|} = 1$$

by equation (3.20)

Now that we know $c = 1$ we obtain equation (3.27).

Next, fix $\eta \in G_{k,n}$ and change to k -dimensional polar coordinates in the following integral over η , keeping in mind that S^{k-1} refers to the intersection of S^{n-1} with η :

$$\begin{aligned} \int_{\eta} |x'|^{n-k} f(x') \, dx' &= \int_{S^{k-1}} \int_0^{\infty} r^{n-k} f(r\theta) r^{k-1} \, dr \, d\theta \\ &= \int_{S^{k-1}} \int_0^{\infty} r^{n-1} f(r\theta) \, dr \, d\theta \\ &= \int_0^{\infty} r^{n-1} \left(\int_{S^{n-1} \cap \eta} f(r\theta) \, d\theta \right) \, dr. \end{aligned} \tag{3.28}$$

We used Fubini's theorem to change the order of integration. This is valid if f is either measurable and nonnegative or if f is integrable. Integrating this over $G_{k,n}$ and using Fubini again gives

$$\int_{G_{k,n}} \int_{\eta} |x'|^{n-k} f(x') \, dx' \, d\eta = \int_0^{\infty} r^{n-1} \left(\int_{G_{k,n}} \int_{S^{n-1} \cap \eta} f(r\theta) \, d\theta \, d\eta \right) \, dr.$$

We can now apply equation (3.27) with the function $f(r\theta)$ in place of $f(\theta)$ to the integral in parentheses to obtain

$$\begin{aligned} \int_{G_{k,n}} \int_{\eta} |x'|^{n-k} f(x') \, dx' \, d\eta &= |G_{k-1,n-1}| \int_0^{\infty} \int_{S^{n-1}} f(r\theta) \, d\theta r^{n-1} \, dr \\ &= |G_{k-1,n-1}| \int_{\mathbb{R}^n} f(x) \, dx \end{aligned}$$

which establishes the first equation of the theorem. In the last step we used the usual polar coordinate substitution on $\mathbb{R}^n$.

To finish the proof we apply the first equation to the case where we have $n - k$ instead of k . Letting ξ be a variable point in $G_{n-k,n}$ and letting x'' be a variable point in ξ we then have

$$|G_{n-k-1,n-1}| \int_{\mathbb{R}^n} f(x) dx = \int_{G_{n-k,n}} \int_{\xi} |x''|^k f(x'') dx'' d\xi. \quad (3.29)$$

Let $F(\xi) = \int_{\xi} |x''|^k f(x'') dx''$. Applying proposition 3.8 to F changes the domain of integration of the right-hand integral in equation (3.29) from $G_{n-k,n}$ to $G_{k,n}$ yielding

$$\int_{G_{n-k,n}} \int_{\xi} |x''|^k f(x'') dx'' d\xi = \int_{G_{k,n}} \int_{\eta^\perp} |x''|^k f(x'') dx'' d\eta.$$

Substituting this result in the preceding equation gives

$$|G_{n-k-1,n-1}| \int_{\mathbb{R}^n} f(x) dx = \int_{G_{k,n}} \int_{\eta^\perp} |x''|^k f(x'') dx'' d\eta.$$

But $|G_{n-k-1,n-1}| = |G_{k,n-1}|$ by equation (3.21) and this finishes the proof of the theorem. ■

Remark 3.10. If we let $k = 1$ then the theorem reads $\int_{G_{1,n}} \int_{\eta} |x'|^{n-1} f(x') dx' d\eta = \int_{\mathbb{R}^n} f(x) dx$, because $|G_{0,n-1}| = 1$. In this case $\eta \in G_{1,n}$ is a line which corresponds to exactly two unit vectors of opposite direction in S^{n-1} . Thus the integration over η can be thought of as the sum of two integrals over rays and the previous equation can be rewritten as $\int_{S^{n-1}} \int_0^\infty r^{n-1} f(r\theta) dr d\theta = \int_{\mathbb{R}^n} f(x) dx$. Therefore, theorem 3.9 specializes to a polar coordinate substitution if $k = 1$ and this explains why the theorem is called the “generalized polar coordinate theorem.”

3.4 The Definition and Basic Properties of the k -Plane Transform and Its Adjoint

The k -plane transform operates on a function by integrating it over k -planes in $\mathbb{R}^n$. Let us consider first the one-plane transform, also called the ***x-ray transform*** or ***John transform***. It is less awkward to use the terminology “*x-ray transform*” and we shall do so in favor of “one-plane transform.” In this case a k -plane is merely a straight line in $\mathbb{R}^n$. We can parametrize such lines by choosing a one-dimensional linear subspace η of $\mathbb{R}^n$, that is, a line through the origin. Then for each point $x'' \in \eta^\perp$ we form the following integral over the line $\eta + x''$

$$\int_{x' \in \eta} f(x' + x'') dx'$$

and consider this integral to be the value of the x-ray transform at the ordered pair (η, x'') . We denote this by $Xf(\eta, x'')$. Occasionally we abuse the notation by writing $Xf(\theta, x'')$ for $Xf(\eta, x'')$, where η is the one-dimensional subspace generated by the unit vector θ . In this case x'' is any vector orthogonal to θ . Also, recall the convention that variables of the form x' live in η while those of the form x'' live in $\eta^\perp$. Also, if η is a variable in the Grassmannian manifold $G_{k,n}$, then we may use $d\eta$, instead of $\nu_{k,n}$ to denote the Haar measure of $G_{k,n}$.

The definition of the x-ray transform is analogous to the definition of the Radon transform (definition 2.28). Indeed in dimension $n = 2$ the x-ray transform and the Radon transform are equivalent; in each case we obtain the line integrals of f over all lines in $\mathbb{R}^2$. The main difference is that the Radon transform specifies a line by giving the direction θ orthogonal to the line being integrated and the distance p of that line from the origin, whereas the x-ray transform specifies the direction of the line, via η , and a point through which that line passes. More precisely, in $\mathbb{R}^2$, given a line η through the origin and a point $x'' \in \eta^\perp$ with $x'' \neq 0$, define

$$\theta = \frac{x''}{|x''|} \in S^{n-1}$$

$$p = |x''|.$$

An easy calculation shows that

$$Xf(\eta, x'') = Rf(\theta, p).$$

Conversely, given θ, p if we define η to be the line through the origin orthogonal to θ and $x'' = p\theta$, then we get the same relationship between the x-ray transform and the Radon transform in $\mathbb{R}^2$. It is then clear that both the x-ray transform X and the Radon transform R allow one to obtain the line integrals of a function f defined on $\mathbb{R}^2$.

On the other hand, the x-ray transform differs in a fundamental way from the Radon transform in higher dimensional spaces.

We define the k -plane transform, denoted by P , as a generalization of the x-ray transform. Since the x-ray transform integrates functions over lines that are 1-planes, then we would want the k -plane transform to integrate functions over k -planes. We therefore define the domain of the k -plane transform Pf of f to be the set of k -planes in $\mathbb{R}^n$. We parametrize these k -planes via the following definition.

Definition 3.11 (affine Grassmannians).

$$\mathcal{G}_{k,n} = \left\{ (\eta, x'') : \eta \in G_{k,n} \text{ and } x'' \in \eta^\perp \right\}$$

Noting that $\mathcal{G}_{k,n} \subset G_{k,n} \times \mathbb{R}^n$, we topologize $\mathcal{G}_{k,n}$ by giving it the induced topology. Then $\mathcal{G}_{k,n}$ is a locally compact topological space and it becomes a measure space by taking the measure induced by the positive linear functional

$$g \mapsto \int_{\eta \in G_{k,n}} \left[\int_{x'' \in \eta^\perp} g(\eta, x'') dx'' \right] d\eta$$

We let $dx''d\eta$ denote the resulting measure.

$\mathcal{G}_{k,n}$ is called an **affine Grassmannian**.

Remark 3.12. $\mathcal{G}_{k,n}$ has a natural one-to-one correspondence with the set of k -planes in $\mathbb{R}^n$. If we define the mapping by $(\eta, x'') \mapsto \eta + x''$, where $\eta \in G_{k,n}$ is a k -dimensional vector space in $\mathbb{R}^n$ and where $x'' \in \eta^\perp$, then the injectivity is an easy consequence of the orthogonality conditions. It is also obvious that the mapping is surjective.

By definition, an affine space is a translation of a linear space. Therefore, affine spaces in $\mathbb{R}^n$ are precisely the k -planes. Since the usual Grassmannian $G_{k,n}$ is the set of all linear spaces of dimension k in $\mathbb{R}^n$, then it is reasonable to call $\mathcal{G}_{k,n}$, which is identified with the set of affine spaces of dimension k in $\mathbb{R}^n$, an affine Grassmannian.

Remark 3.13. Those readers familiar with fiber bundles will recognize $\mathcal{G}_{k,n}$ as a vector bundle over the Grassmannian $G_{k,n}$ with fibers isomorphic to $\mathbb{R}^{n-k}$.

Remark 3.14. The motion group $M(n)$ of $\mathbb{R}^n$ consists of all isometries of $\mathbb{R}^n$. In much the same way that we proved that $G_{k,n} = O(n)/O(k) \times O(n-k)$, it can be proved that $\mathcal{G}_{k,n} = M(n)/M(k) \times O(n-k)$. Since it is obvious that $M(n) = O(n) \oplus \mathbb{R}^n$, it follows that the dimension of $\mathcal{G}_{k,n}$ as a differentiable manifold can be computed to be $\dim \mathcal{G}_{k,n} = (k+1)(n-k)$ (see remark 3.7).

Remark 3.15. In particular, the manifold $\mathcal{G}_{n-1,n}$ of hyperplanes can be identified with the homogeneous space $M(n)/\mathbb{Z}_2 \times M(n-1)$. This manifold has dimension n .

The Grassmannian $G_{k,n}$ consists of k -dimensional linear subspaces of $\mathbb{R}^n$. Therefore, if we use $\eta \in G_{k,n}$ as a variable for a generic k -dimensional linear space, then $\eta + x''$ is a generic k -plane. Integrating a function over a generic k -plane results in the k -plane transform, which we now precisely define.

Definition 3.16 (k -plane transform). Let f be a measurable function defined on $\mathbb{R}^n$. Then for $(\eta, x'') \in \mathcal{G}_{k,n}$ we define the k -plane transform of f by

$$Pf(\eta, x'') = \int_{x' \in \eta} f(x' + x'') dx',$$

where dx' denotes the induced Lebesgue measure on the k space η .

We should be aware that the integral defining $Pf(\eta, x'')$ may not exist. However, if f is a nonnegative measurable function, then the integral will exist as an extended real number (i.e., the value will be either finite or ∞).

The 1-plane transform is called the **x -ray transform** and is denoted either by P (if it is understood that $k = 1$) or by X . We allow the abuse of notation that arises by writing $Xf(\theta, x'')$ for $Xf(\eta, x'')$, where η is the one-dimensional subspace generated by the unit vector θ . In this case x'' is any vector orthogonal to θ .

We also let

$$P_\eta f(x'') = Pf(\eta, x'')$$

and we call $P_\eta f$ the **k -plane projection** (in the “direction” η).

In the case of the x-ray transform, which is the k -plane transform with $k = 1$, we use the notation $X_\theta f$ to denote the **x-ray projection** $P_\eta f$, where η is the line through the origin in the direction θ .

There is clearly a different k -plane transform for each value of $k = 1, \dots, n - 1$. The value of k will be apparent in any discussion and it is omitted from the notation.

Remark 3.17. The k -plane transform is sometimes called the **Radon–John transform**. We already know that Johann Radon developed the integral transform which integrates over $n - 1$ planes [508], whereas Fritz John developed the x-ray transform which integrates over 1 planes [324]. Some authors use the term “John transform” for “x-ray transform” (see Gel’fand, Gindikin, and Graev [189]). The terminology Radon–John transform is therefore indicative of transforms that integrate over k -planes with $0 < k < n$.

Remark 3.18. Whenever we deal with k -planes we always assume that k is an integer in the range $0 < k < n$, where n is the dimension of the ambient space.

Some authors, for example, Keinert [348], call this transform the “parallel k -plane transform” because the integration is performed over parallel planes. The reason for the choice of the letter P to denote the k -plane transform appears to be related to this terminology. There is another transform called the divergent k -plane transform in which integration is performed on planes diverging from a common point. Other authors use the letter L to denote the k -plane transform (compare, Solmon [569, 570]) or the symbol $\mathcal{R}_k$ (Calderón [74]), and there are still more variations on this theme.

At this time we also define the “backprojection” operator for the k -plane transform.

Definition 3.19. If $\eta \in G_{k,n}$, then proj_η denotes the orthogonal projection operator $\text{proj}_\eta : \mathbb{R}^n \rightarrow \eta$ from $\mathbb{R}^n$ with the usual inner product to the subspace η .

Definition 3.20 (Backprojection: formal adjoint). Let g be a measurable function defined on $G_{k,n}$. For $x \in \mathbb{R}^n$, define

$$P^\# g(x) = \int_{G_{k,n}} g(\eta, \text{proj}_{\eta^\perp}(x)) d\eta$$

$P^\#$ is called the **formal adjoint** of the k -plane transform. It is also called the **backprojection operator** or the **dual k -plane transform**.

Remark 3.21. See theorems 3.29 and 3.104 for a justification of the terminology “formal adjoint.” In chapters 1 and 2 we explained why the formal adjoint is called a backprojection operator.

A simple but useful property of the k -plane transform is its behavior under translations (sometimes called shifts). Recall that the **translation** f_a of a function f by a vector a is the function whose value at x is $f(x - a)$.

Theorem 3.22 (Translation theorem for the k -plane transform). *If $a \in \mathbb{R}^n$, $\eta \in G_{k,n}$ and $x'' \in \eta^\perp$, then*

$$Pf_a(\eta, x'') = Pf(\eta, x'' - \text{proj}_{\eta^\perp} a).$$

Proof. Let $a' = \text{proj}_\eta a$, $a'' = \text{proj}_{\eta^\perp} a$. Then $a = a' + a''$ and

$$\begin{aligned} Pf_a(\eta, x'') &= \int_{\eta} f_a(x' + x'') dx' = \int_{\eta} f((x' - a') + (x'' - a'')) dx' \\ &= \int_{\eta} f(x' + (x'' - a'')) dx' \\ &= Pf(\eta, x'' - \text{proj}_{\eta^\perp} a) \end{aligned}$$

since integration on the k -space η is translation invariant. ■

The next theorem follows directly from the definitions of the k -plane transform and its formal adjoint.

Theorem 3.23. *The k -plane transform is a linear transform on any linear space of functions on $\mathbb{R}^n$ for which it is defined. The same is true for the formal adjoint on any linear space of functions on $G_{k,n}$ for which it is defined.*

More precise results of the nature of theorem 3.23, such as corollary 3.25 which follows the next result, are possible. But first we need the next very useful result, which is essentially a generalized version of Fubini's theorem.

Proposition 3.24. *Let f be a measurable function on $\mathbb{R}^n$, let $\eta \in G_{k,n}$ be a k -space, and let ρ be a measurable function of one variable. If $y'' \in \eta^\perp$ is such that the function $\rho(\langle x'', y'' \rangle) Pf(\eta, x'')$ is integrable on $\eta^\perp$, then $\rho(\langle x, y'' \rangle) f(x)$ is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \rho(\langle x, y'' \rangle) f(x) dx = \int_{\eta^\perp} \rho(\langle x'', y'' \rangle) Pf(\eta, x'') dx''. \quad (3.30)$$

Conversely, if for a fixed $y'' \in \eta^\perp$ the function $\rho(\langle x, y'' \rangle) f(x)$ is integrable on $\mathbb{R}^n$, then the function $\rho(\langle x'', y'' \rangle) Pf(\eta, x'')$ is integrable on $\eta^\perp$ and equation (3.30) holds.

In the case $k = n - 1$, the k -plane transform is essentially the Radon transform and we can take $y'' = \theta$ and $x'' = t\theta$. The formula then becomes

$$\int_{\mathbb{R}^n} \rho(\langle x, \theta \rangle) f(x) dx = \int_{\eta^\perp} \rho(t) Rf(\theta, t) dt. \quad (3.31)$$

Proof. Since $\mathbb{R}^n = \eta \oplus \eta^\perp$ it follows that any $x \in \mathbb{R}^n$ can be uniquely written in the form $x = x' + x''$, where $x' \in \eta$ and $x'' \in \eta^\perp$. By the orthogonality we see that $\rho(\langle x, y'' \rangle) = \rho(\langle x'', y'' \rangle)$. Fubini's theorem applies in the coordinate system defined by the decomposition: $\mathbb{R}^n = \eta \oplus \eta^\perp$.

First, if we assume that $\rho(\langle x'', y'' \rangle)Pf(\eta, x'')$ is integrable on $\eta^\perp$, then the integrability of $\rho(\langle x, y'' \rangle)Pf(\eta, x'')$ on $\mathbb{R}^n$ follows from Fubini. Furthermore, we have:

$$\begin{aligned} \int_{\eta^\perp} \rho(\langle x'', y'' \rangle) Pf(\eta, x'') dx'' &= \int_{\eta^\perp} \rho(\langle x'', y'' \rangle) \left(\int_{\eta} f(x' + x'') dx' \right) dx'' \\ &= \int_{\eta^\perp} \int_{\eta} \rho(\langle x, y'' \rangle) f(x) dx' dx'' \\ &= \int_{\mathbb{R}^n} \rho(\langle x, y'' \rangle) f(x) dx. \end{aligned}$$

Conversely, if for a fixed $y'' \in \eta^\perp$ the function $\rho(\langle x, y'' \rangle)f(x)$ is integrable on $\mathbb{R}^n$, then Fubini implies that the function $\rho(\langle x'', y'' \rangle)Pf(\eta, x'')$ is integrable on $\eta^\perp$ and that the previous formula holds. ■

Corollary 3.25. *If $f \in L^1(\mathbb{R}^n)$, then Pf is integrable on $\mathcal{G}_{k,n}$. In particular, the integral defining $Pf(\eta, x'')$ exists as a Lebesgue integral almost everywhere with respect to the measure $dx'' d\eta$ of $\mathcal{G}_{k,n}$. Furthermore, the k -plane transform $P : L^1(\mathbb{R}^n) \rightarrow L^1(\mathcal{G}_{k,n})$ is a bounded linear transformation and*

$$\|Pf\|_{L^1(\mathcal{G}_{k,n})} \leq |G_{k,n}| \|f\|_{L^1(\mathbb{R}^n)} \quad (3.32)$$

Also, if f is integrable on $\mathbb{R}^n$ and $\eta \in G_{k,n}$ is a k -space, then

$$\int_{\mathbb{R}^n} f(x) dx = \int_{x'' \in \eta^\perp} Pf(\eta, x'') dx''. \quad (3.33)$$

This can also be expressed as

$$\int_{\mathbb{R}^n} f(x) dx = \int_{x'' \in \eta^\perp} \int_{x' \in \eta} f(x' + x'') dx' dx'' \quad (3.34)$$

Proof. First we see that equation (3.33) is an immediate corollary of proposition 3.24 upon taking $\rho(t) \equiv 1$.

The estimate (3.32) is established as follows:

$$\|Pf\|_{L^1(\mathcal{G}_{k,n})} = \int_{\eta \in G_{k,n}} \left[\int_{x'' \in \eta^\perp} |Pf(\eta, x'')| dx'' \right] d\eta$$

$$\begin{aligned}
&\leq \int_{\eta \in G_{k,n}} \left[\int_{x'' \in \eta^\perp} P |f|(\eta, x'') dx'' \right] d\eta \\
&= \int_{\eta \in G_{k,n}} \left[\int_{\mathbb{R}^n} |f(x)| dx \right] d\eta \quad (\text{by (3.33)}) \\
&= |G_{k,n}| \|f\|_{L^1(\mathbb{R}^n)}
\end{aligned}$$

■

Remark 3.26. Later on, in theorem 3.35, we will see that we can extend this result to certain nonintegrable functions.

Another corollary to proposition 3.24 is the slice-projection theorem that relates the k -plane transform to the Fourier transform. This result is also known as the central slice theorem.

We define a Fourier transform on $\mathcal{G}_{k,n}$ in the following way. If (η, ξ'') is in $\mathcal{G}_{k,n}$ and if g is an L^1 function in the second variable, then

$$\widehat{g}(\eta, \xi'') = (2\pi)^{-\frac{n-k}{2}} \int_{\eta^\perp} e^{-i\langle x'', \xi'' \rangle} P f(\eta, x'') dx''$$

The integration is over the $n - k$ -space $\eta^\perp$ which accounts for the factor $(2\pi)^{-\frac{n-k}{2}}$. This definition extends to functions that are L^2 in the second variable.

Theorem 3.27 (Slice-projection theorem). *If $f \in L^1(\mathbb{R}^n)$ and $0 < k < n$, then*

$$\widehat{P f}(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'') \quad \text{for } \xi'' \in \eta^\perp$$

Proof. Using proposition 3.24 with $\rho(t) = e^{-it}$ and $y'' = \xi''$, we obtain

$$\begin{aligned}
\widehat{P f}(\eta, \xi'') &= (2\pi)^{-\frac{n-k}{2}} \int_{\eta^\perp} e^{-i\langle x'', \xi'' \rangle} P f(\eta, x'') dx'' \\
&= (2\pi)^{-\frac{n-k}{2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi'' \rangle} f(x) dx \\
&= (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')
\end{aligned}$$

■

It is obvious from this result that the k -plane transform is injective on $L^1(\mathbb{R}^n)$, compare theorem 3.131.

The slice-projection theorem can also be stated in the form $\widehat{P_\eta f}(\xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')$, which means that the k -dimensional Fourier transform of the k -plane projection $P_\eta f$ is a constant multiple of the n -dimensional Fourier transform of f restricted to the “slice” $\eta^\perp$ of $\mathbb{R}^n$.

The Radon transform defined in chapter 2 integrates functions over hyperplanes in $\mathbb{R}^n$. The same is true for the k -plane transform with $k = n - 1$. The following result describes the connection between the $(n - 1)$ -plane transform and the Radon transform.

Theorem 3.28. *If R is the Radon transform and P is the $(n - 1)$ -plane transform, then for any $\theta \in S^{n-1}$ and any $s \in \mathbb{R}$ we have*

$$Pf(\theta^\perp, s\theta) = Rf(\theta, s). \quad (3.35)$$

For a function g defined on $G_{n-1,n}$ define a function G on the cylinder $S^{n-1} \times \mathbb{R}$ in the following way:

$$G(\theta, s) = g(\theta^\perp, s\theta) \quad (3.36)$$

Then

$$P^\# g(x) = \frac{1}{2} R^\# G(x).$$

Proof. For each $\theta \in S^{n-1}$, we define η to be the hyperplane $\theta^\perp$ and we define $x'' = s\theta \in \eta^\perp$. Therefore, we can express the $(n - 1)$ -plane transform as $Pf(\theta^\perp, s\theta) = \int_{x' \in \theta^\perp} f(x' + s\theta) dx'$, which, according to definition 2.28, is the same as $Rf(\theta, s)$.

The relation between the adjoint operators is established as follows. The map $\Phi : S^{n-1} \rightarrow G_{n-1,n}$ defined by $\theta \mapsto \theta^\perp$ is continuous. Therefore, if we define the functional λ on $C^0(G_{n-1,n})$ by $\lambda(h) = \int_{S^{n-1}} h(\Phi(\theta)) d\theta$, then it is easy to check that λ is a continuous linear functional on the homogeneous space $G_{n-1,n}$. Furthermore, this functional is invariant with respect to the orthogonal group, so it must be a constant multiple of the Haar measure that we defined on the Grassmannian $G_{n-1,n}$. Hence, a constant c exists such that

$$\lambda(h) = \int_{S^{n-1}} h(\Phi(\theta)) d\theta = c \int_{G_{n-1,n}} h(\eta) d\eta. \quad (3.37)$$

If we take h to be identically 1 on $G_{n-1,n}$, then $h(\Phi(\theta))$ is identically 1 on S^{n-1} . Using this in equation (3.37) yields $|S^{n-1}| = c|G_{n-1,n}|$. But $|G_{n-1,n}| = \frac{1}{2}|S^{n-1}|$ (see the definition of Haar measure on Grassmannians in section 3.3.4 so $c = 2$. Equation (3.37) can therefore be written as

$$\int_{G_{n-1,n}} h(\eta) d\eta = \frac{1}{2} \int_{S^{n-1}} h(\Phi(\theta)) d\theta. \quad (3.38)$$

We now apply this result to the function $h(\eta) = g(\eta, \text{proj}_{\eta^\perp}(x))$. We then have $h(\Phi(\theta)) = g(\theta^\perp, \text{proj}_{(\theta^\perp)^\perp}(x))$. It is easy to see that $\text{proj}_{(\theta^\perp)^\perp}(x) = \langle x, \theta \rangle \theta$ so $h(\eta) =$

$g(\theta^\perp, \langle x, \theta \rangle \theta)$ and hence

$$\begin{aligned} P^\# g(x) &= \int_{G_{n-1,n}} g(\eta, \text{proj}_{\eta^\perp}(x)) d\eta \\ &= \frac{1}{2} \int_{S^{n-1}} g(\theta^\perp, \langle x, \theta \rangle \theta) d\theta \\ &= \frac{1}{2} R^\# G(x). \end{aligned}$$

■

In chapter 2, we saw that the backprojection operator $R^\#$ played a key role in both the exact and approximate inversion of the Radon transform. We now begin to investigate a similar role for $P^\#$ in relation to the k -plane transform. The first result has an almost obvious proof but is of fundamental importance.

First, we define a formal inner product for functions on $\mathbb{R}^n$ by

$$\langle f_1, f_2 \rangle_{\mathbb{R}^n} = \int_{\mathbb{R}^n} f_1(x) \overline{f_2(x)} dx$$

where f_1 and f_2 are either Lebesgue integrable or Lebesgue measurable, nonnegative functions on $\mathbb{R}^n$. This is a formal inner product, because it is not defined on $L^2(\mathbb{R}^n)$ but only for certain functions. Furthermore it is quite possible that this inner product can be infinite. This is the reason that we use the qualifier “formal” for this “inner product.”

In a similar way we define a formal inner product for integrable or nonnegative measurable functions on $\mathcal{G}_{k,n}$:

$$\langle g_1, g_2 \rangle_{G_{k,n}} = \int_{\eta \in G_{k,n}} \int_{x'' \in \eta^\perp} g_1(\eta, x'') \overline{g_2(\eta, x'')} dx'' d\eta$$

If there is no confusion as to the proper domain, we may drop the subscript on the inner product notation, e.g., $\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_{\mathbb{R}^n}$.

We can now prove that $P^\#$ is the formal adjoint of P in the sense that $\langle f, P^\# g \rangle = \langle Pf, g \rangle$, with the full realization that both sides of this equation may be infinite in many cases. More precisely we have:

Theorem 3.29 (Formal adjoint theorem). *Let f be an integrable or a nonnegative measurable function on $\mathbb{R}^n$ and let g be a nonnegative measurable function on $\mathcal{G}_{k,n}$. Then*

$$\langle f, P^\# g \rangle_{\mathbb{R}^n} = \langle Pf, g \rangle_{G_{k,n}}$$

Proof.

$$\begin{aligned} \langle Pf, g \rangle_{G_{k,n}} &= \int_{G_{k,n}} \int_{\eta^\perp} Pf(\eta, x'') \overline{g(\eta, x'')} dx'' d\eta \\ &= \int_{G_{k,n}} \int_{\eta^\perp} \left[\int_{x' \in \eta} f(x' + x'') dx' \right] \overline{g(\eta, x'')} dx'' d\eta \end{aligned}$$

The hypothesis that f is integrable or nonnegative and measurable allows the use of Fubini's theorem. Make the substitution $x = x' + x''$. By Fubini's theorem we obtain $dx = dx' dx''$ and the fact that we can interchange the order of integration. When we do so the integrations over η and $\eta^\perp$ coalesce into a single integral over $\mathbb{R}^n$. We also have $x'' = \text{proj}_{\eta^\perp}(x)$, so, after changing the order of integration, the preceding integral can be written as

$$\begin{aligned} \int_{\mathbb{R}^n} f(x) \left(\int_{G_{k,n}} \overline{g(\eta, \text{proj}_{\eta^\perp}(x))} d\eta \right) dx &= \int_{\mathbb{R}^n} f(x) \overline{P^\# g(x)} dx \\ &= \langle f, P^\# g \rangle_{\mathbb{R}^n} \end{aligned}$$

■

Corollary 3.30. *If g is integrable on $G_{k,n}$, then $P^\# g$ is locally integrable on $\mathbb{R}^n$.*

Proof. Let x_0 be an arbitrary point in $\mathbb{R}^n$, let B be the closed ball of radius 1 centered at x_0 , and let χ be the characteristic function of B . For any $x'' \in \eta^\perp$, the intersection of $\eta + x''$ with B is either a k -dimensional ball of radius $r \leq 1$ or the empty set. Therefore, $P\chi(\eta, x'')$, which is the integral of the characteristic function of B over $\eta + x''$, is either 0 or the volume of a ball of radius $r \leq 1$ and we therefore obtain $P\chi(\eta, x'') \leq |B^k(1)|$ (recall that $|B^k(r)|$ denotes the k -dimensional volume of a ball of radius r in $\mathbb{R}^k$). From this and the formal adjoint theorem applied to the functions χ and g we get

$$\begin{aligned} \int_B |P^\# g(x)| dx &= \int_{\mathbb{R}^n} \chi(x) |P^\# g(x)| dx \\ &\leq \langle \chi, P^\# |g| \rangle_{\mathbb{R}^n} = \langle P\chi, |g| \rangle_{G_{k,n}} \\ &= \int_{G_{k,n}} \int_{\eta^\perp} P\chi(\eta, x'') |g(\eta, x'')| dx'' d\eta \\ &\leq |B^k(1)| \int_{G_{k,n}} \int_{\eta^\perp} |g(\eta, x'')| dx'' d\eta \\ &= |B^k(1)| \|g\|_{L^1(G_{k,n})} \end{aligned}$$

The last term is finite because $g \in L^1(G_{k,n})$ by hypothesis. Because x_0 was arbitrary, this proves that $P^\# g$ is locally integrable on $\mathbb{R}^n$. ■

We close this section with formulas for computing the k -plane transform and the backprojection of radial functions. A **radial function** is one of the form $f(|x|)$ where f is a function of a single real variable. Theorem 3.31 is a generalization of theorem 2.43 of chapter 2 which showed that the Radon transform of a radial function is radial. Here we also get useful formulas for the k -plane transform and its adjoint. This theorem is well known in the field. Except for minor variations, the statement and proof of this version is due to Rubin [538].

Theorem 3.31. *Let $F(x) = f(|x|)$ be a radial function and let P be the k -plane transform. Then for any $(\eta, x'') \in \mathcal{G}_{k,n}$ we have*

$$PF(\eta, x'') = |S^{k-1}| \int_{|x''|}^{\infty} f(r) \left(r^2 - |x''|^2 \right)^{\frac{k}{2}-1} r dr \quad (3.39)$$

Also if $G(\eta, x'') = g(|x''|)$, and $x \in \mathbb{R}^n$, then

$$P^\# G(x) = \frac{|S^{k-1}| |S^{n-k-1}|}{|S^{n-1}|} \frac{1}{|x|^{n-2}} \int_0^{|x|} g(s) \left(|x|^2 - s^2 \right)^{\frac{k}{2}-1} s^{n-k-1} ds \quad (3.40)$$

Proof. We can introduce polar coordinates in the k -plane determined by (η, x'') as follows. Any point of this k -plane has the form $x'' + t\theta$ for some θ in the unit sphere of η and some $t \geq 0$. The unit sphere of η is the same as $\eta \cap S^{n-1}$ and since η is k -dimensional, this sphere is $k-1$ -dimensional and hence $|\eta \cap S^{n-1}| = |S^{k-1}|$. Using the k -dimensional polar coordinates centered at x'' on this k -plane and keeping in mind that S^{k-1} is $\eta \cap S^{n-1}$ we get

$$\begin{aligned} PF(\eta, x'') &= \int_{x' \in \eta} F(x' + x'') dx' \\ &= \int_0^\infty \int_{\eta \cap S^{n-1}} F(t\theta + x'') t^{k-1} d\theta dt = \int_0^\infty \int_{\eta \cap S^{n-1}} f(|t\theta + x''|) t^{k-1} d\theta dt \end{aligned}$$

Since $x'' \perp t\theta$, then $|t\theta + x''| = \sqrt{t^2 + |x''|^2}$ so we get

$$\begin{aligned} PF(\eta, x'') &= \int_{\eta \cap S^{n-1}} d\theta \int_0^\infty f(\sqrt{t^2 + |x''|^2}) t^{k-1} dt \\ &= |S^{k-1}| \int_0^\infty f(\sqrt{t^2 + |x''|^2}) t^{k-1} dt. \end{aligned}$$

On making the change of variable $r = \sqrt{t^2 + |x''|^2}$ the proof of equation (3.39) is finished.

We give just a sketch of the proof of equation (3.40). By the definition of the adjoint of the k -plane transform we have $P^\# G(x) = \int_{G_{k,n}} g(|\text{proj}_{\eta^\perp}(x)|) d\eta$. In this expression we are projecting over all possible $n - k$ -dimensional subspaces $\eta^\perp$ of $\mathbb{R}^n$. The set of these subspaces is generated by all possible rotations of the specific $n - k$ -dimensional subspace $\mathbb{R}^{n-k}$. In other words we are projecting x onto all subspaces of the form $\gamma \mathbb{R}^{n-k}$ as γ ranges over $SO(n)$, the special orthogonal group.

In the proof of the generalized polar coordinate theorem we were able to prove the equality of two integrals by defining a Haar measure via a continuous linear functional. A similar method of proof shows that $\int_{G_{k,n}} g(|\text{proj}_{\eta^\perp}(x)|) d\eta = \int_{SO(n)} g(|\text{proj}_{\gamma \mathbb{R}^{n-k}}(x)|) d\gamma$. The vector obtained by projecting x onto the rotation $\gamma \mathbb{R}^{n-k}$ of $\mathbb{R}^{n-k}$ has the same length as the vector obtained by rotating x and projecting onto $\mathbb{R}^{n-k}$. All such rotations can be performed by taking a unit vector and multiplying by $|x|$. The resulting lengths are $|\text{proj}_{\mathbb{R}^{n-k}} |x| \theta|$, for $\theta \in S^{n-1}$. By the method mentioned previously we can also show that $\int_{SO(n)} g(|\text{proj}_{\gamma \mathbb{R}^{n-k}}(x)|) d\gamma = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} g(|\text{proj}_{\mathbb{R}^{n-k}} |x| \theta|) d\theta$. Combining these results gives

$$P^\# G(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} g(|\text{proj}_{\mathbb{R}^{n-k}} |x| \theta|) d\theta.$$

We now perform a change of variable to bispherical coordinates in the last integral. (see discussion of bispherical coordinates, in section 3.2 esp. eqn. (3.9)) We let $\theta = \cos(\psi)a + \sin(\psi)b$, where $a \in S^{k-1}$, $b \in S^{n-k-1}$. In particular, a is orthogonal to $\mathbb{R}^{n-k}$ so

$$\text{proj}_{\mathbb{R}^{n-k}} |x| \theta = |x| \sin(\psi) b$$

and since b has unit length, the previous equation, when transformed to bispherical coordinates, becomes

$$\begin{aligned} P^\# G(x) &= \frac{1}{|S^{n-1}|} \int_{b \in S^{n-k-1}} \int_{a \in S^{k-1}} \\ &\quad \times \int_{\psi=0}^{\frac{\pi}{2}} g(|x| \sin(\psi)) \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) d\psi da db \\ &= \frac{1}{|S^{n-1}|} \left(\int_{S^{n-k-1}} db \right) \left(\int_{S^{k-1}} da \right) \\ &\quad \times \left(\int_0^{\frac{\pi}{2}} g(|x| \sin(\psi)) \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) d\psi \right) \end{aligned}$$

(equation (3.9)). Since $\int_{S^{n-k-1}} db = |S^{n-k-1}|$, and $\int_{S^{k-1}} da = |S^{k-1}|$, then we get

$$P^\# G(x) = \frac{|S^{k-1}| |S^{n-k-1}|}{|S^{n-1}|} \int_0^{\frac{\pi}{2}} g(|x| \sin(\psi)) \sin^{n-k-1}(\psi) \cos^{k-1}(\psi) d\psi.$$

Finally, after making the change of variable $s = |x| \sin(\psi)$ we obtain equation (3.40). ■

3.5 Lower-Dimensional Integrability of the k -Plane Transform and an Extension of Its Domain

If $p = 1$, and $0 < k < n$, then we have already established in corollary 3.25 that the k -plane transform Pf exists almost everywhere for any $f \in L^p(\mathbb{R}^n)$. The situation is much more subtle for L^p if $p > 1$ as the next two results show. These results are from Smith and Solmon [565] and Solmon [570].

Theorem 3.32. *If $1 < p \leq \infty$, then there exists a nonnegative function $f \in L^p(\mathbb{R}^n)$ such that f is not integrable over any k -plane of dimension $k \geq \frac{n}{p}$.*

Proof. Define the radial function f by

$$f(x) = |x|^{-\frac{n}{p}} \frac{1}{\ln(|x|)} \quad \text{if } |x| \geq e, \quad f(x) = 0, \text{ otherwise.}$$

Then $f \in L^p(\mathbb{R}^n)$ as the next computation shows.

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x)|^p dx &= |S^{n-1}| \int_e^\infty \frac{1}{r (\ln(r))^p} dr \\ &= \int_1^\infty \frac{1}{u^p} du. \end{aligned}$$

This integral is finite since $p \geq \frac{n}{k} > 1$. Therefore, F is a nonnegative function in $L^p(\mathbb{R}^n)$.

We now show that F is not integrable on any k -plane with $k \geq \frac{n}{p}$. By theorem 3.31, for $|x''| > e$,

$$\begin{aligned} Pf(\eta, x'') &= |S^{k-1}| \int_{|x''|}^\infty r^{-\frac{n}{p}} \frac{1}{\ln(r)} \left(r^2 - |x''|^2\right)^{\frac{k}{2}-1} r dr \\ &\geq c \int_{|x''|}^\infty r^{k-\frac{n}{p}-1} \frac{1}{\ln(r)} dr = c \int_{\ln|x''|}^\infty \frac{e^{\left(k-\frac{n}{p}\right)u}}{u} du. \end{aligned}$$

But this last integral is clearly infinite if $k \geq \frac{n}{p}$. A similar calculation holds if $|x''| < e$. Since $Pf(\eta, x'')$ is the integral of f over a generic k -plane, f is not integrable over any k -plane of dimension $k \geq \frac{n}{p}$. ■

Maybe this theorem does not seem so amazing because we have no reason to expect that an L^p function would be integrable on a k -plane. However, Smith and Solmon [565] proved the remarkable result that any square integrable function is integrable on almost any k -plane with $k < \frac{n}{2}$. In other words, square integrable functions are integrable on k -planes of sufficiently low dimensionality. Therefore, we can call this result the “theorem on lower-dimensional integrability.” Solmon [570] generalized this result as follows.

Theorem 3.33 (Lower-dimensional integrability). *If $p \geq 1$ and if k satisfies the inequality $0 < k < \frac{n}{p}$, then any $f \in L^p(\mathbb{R}^n)$ is integrable over almost every k -plane.*

The theorem on lower-dimensional integrability is a direct consequence of the next result, which gives a large class of nonintegrable functions f for which the k -plane transform of f exists as a finite Lebesgue integral almost everywhere on $\mathcal{G}_{k,n}$. This result is referred to as “extension of the domain of the k plane transform.” We give the proof of the lower-dimensional integrability theorem soon thereafter.

The extension of the domain of the k -plane transform is due to Solmon ([570]) (compare, Calderón [74]). The proof given here, which is much simpler than Solmon’s, is from Rubin [538], as is the next lemma which we require for the proof. Rubin [538] also has some interesting generalizations of this idea.

Lemma 3.34. *Let $\beta > n - k$. In this lemma and the following theorem we let $\psi(r) = \int_0^1 (1 + rt)^{-\beta} (1 - t^2)^{\frac{k}{2}-1} t^{n-k-1} dt$. Then there exists a constant c independent of β, k, n such that $|\psi(r)| \leq c(1 + r)^{k-n}$.*

Proof(sketch). It is easy to check that ψ is continuous on $[0, \infty)$. Therefore it suffices to prove this estimate for sufficiently large r . In fact, for sufficiently large r , this estimate is equivalent to the estimate $|\psi(r)| \leq c r^{k-n}$. This estimate follows from the asymptotic behavior of the hypergeometric function. A direct argument can be given by breaking the integral defining ψ into suitable pieces. For completeness, we have provided the complete proof in the appendix, section 3.14. ■

Theorem 3.35 (Existence theorem for the k -plane transform). *Let P be the k -plane transform.*

Then $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ is a sufficient condition for Pf to exist almost everywhere on $\mathcal{G}_{k,n}$.

This means that for almost every k -dimensional linear space η in $G_{k,n}$ and for almost every $x'' \in \eta^\perp$, the Lebesgue integral defining $Pf(\eta, x'')$ is finite.

Furthermore, if β is any real number with $\beta > n - k$, then the k -plane transform P is a bounded linear operator

$$P : L^1\left(\mathbb{R}^n, (1 + |x|)^{k-n} dx\right) \rightarrow L^1\left(\mathcal{G}_{k,n}, (1 + |x''|)^\beta dx'' d\eta\right). \quad (3.41)$$

Proof. Choose any real number β with $\beta > n - k$. For $\eta \in G_{k,n}$ and $x'' \in \eta^\perp$, let $u(\eta, x'') = (1 + |x''|)^{-\beta}$. We can now apply theorem 3.31 to get

$$P^\# u(x) = c \cdot |x|^{2-n} \int_0^{|x|} (1+s)^{-\beta} \left(|x|^2 - s^2\right)^{\frac{k}{2}-1} s^{n-k-1} ds.$$

If we make the substitution $s = |x| t$, then this simplifies to

$$\begin{aligned} P^\# u(x) &= c \cdot \int_0^1 (1+|x|t)^{-\beta} \left(1-t^2\right)^{\frac{k}{2}-1} t^{n-k-1} dt \\ &= c \cdot \psi(|x|) \end{aligned}$$

where ψ is the function defined in lemma 3.34.

We now apply the formal adjoint theorem $\langle f, P^\# u \rangle_{\mathbb{R}^n} = \langle Pf, u \rangle_{G_{k,n}}$ to obtain

$$\int_{G_{k,n}} Pf(\eta, x'') (1 + |x''|)^{-\beta} dx'' d\eta = c \int_{\mathbb{R}^n} f(x) \psi(|x|) dx.$$

By breaking f into its positive and negative parts and by using lemma 3.34 we obtain

$$\left| \int_{G_{k,n}} Pf(\eta, x'') (1 + |x''|)^{-\beta} dx'' d\eta \right| \leq c \int_{\mathbb{R}^n} |f(x)| (1 + |x|)^{k-n} dx$$

This proves the boundedness of the linear operator $P : L^1(\mathbb{R}^n, (1 + |x|)^{k-n} dx) \rightarrow L^1(G_{k,n}, (1 + |x''|)^\beta dx'' d\eta)$. An immediate consequence is that anytime $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, then the integral on the right of the preceding inequality is finite. Therefore, the integral on the left is finite and Fubini's theorem implies that $Pf(\eta, x'')$ exists for almost every η and x'' . ■

There is a partial converse to this theorem:

Theorem 3.36. *A nonnegative measurable function f defined on $\mathbb{R}^n$ has the property that Pf exists almost everywhere and is locally integrable on $G_{k,n}$, if and only if $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$.*

The demonstration of this result requires some facts on Riesz potentials. Therefore, we postpone its proof until section 3.7 where Riesz potentials are introduced. The proof is at the end of section 3.7.

We now prove the lower-dimensional integrability theorem whose proof was postponed. This theorem states: If $p \geq 1$ and if k satisfies the inequality $0 < k < \frac{n}{p}$, then any $f \in L^p(\mathbb{R}^n)$ is integrable over almost every k -plane.

In the proof p' denotes the conjugate index of p . We have the relation $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof of theorem 3.33. It is easy to check that $k < \frac{n}{p} \implies (k - n) p' < -n$. Hence, $(1 + |x|)^{k-n} \in L^{p'}(\mathbb{R}^n)$ and Hölder's inequality then gives $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, since $f \in L^p$. By the preceding theorem we get the desired result. ■

Remark 3.37. By the lower-dimensional integrability theorem and theorem 3.32 we see that any $f \in L^p(\mathbb{R}^n)$ is integrable over almost every k -plane if and only if $1 < p < \frac{k}{n}$.

Remark 3.38. Theorem 3.35 gives an extension of the domain of the k -plane transform from L^1 functions to a larger class. We will give a different extension of the k -plane transform in section 3.9 later in this chapter.

3.6 An Easy Inversion Formula for the k -Plane Transform

The k -plane transform P is a linear operator. An interesting question is whether P has a left inverse. If such an inverse exists it means that knowing the k -dimensional x -rays of a function f , that is, knowing Pf , allows one to recover f by applying the inverse to Pf . Therefore, the question of the invertibility of the k -plane transform is of great theoretical and practical importance.

In this section we obtain an inversion formula for the k -plane transform which applies to certain integrable functions. In particular, it applies to rapidly decreasing functions. We then apply this result to get more specific inversion formulas for the Radon transforms of smooth functions.

The inverse Fourier transform of functions of the form $|\xi|^\alpha \widehat{f}(\xi)$ plays an important role in the inversion of the k -plane transform, so we define an operator which formalizes the idea. This operator was first introduced by Calderón and Zygmund [76].

Definition 3.39 (Lambda operator). For any real number α and for any tempered distribution u for which $|\xi|^\alpha \widehat{u}(\xi)$ is also a tempered distribution, define $\Lambda^\alpha u = \mathcal{F}^{-1}(|\xi|^\alpha \widehat{u}(\xi))$.

Remark 3.40. Some facts about tempered distributions were reviewed in section 3.2. In particular, tempered distributions have Fourier transforms that are also tempered distributions. Also functions in the class L^p , $1 \leq p \leq \infty$ are tempered distributions.

Remark 3.41. From the previous remark, $\Lambda^\alpha f$ is well defined when $|\xi|^\alpha \widehat{f}(\xi) \in L^1$. In particular, $\Lambda^\alpha f$ is defined for any f in the Schwartz class $\mathcal{S}$.

We will have occasion to use the lambda operator on functions of the form $g(\eta, x'')$, where $(\eta, x'') \in \mathcal{G}_{k,n}$. In this case apply Λ^α with respect to the second variable only. Since the lambda operator is defined as an inverse Fourier transform, then in this case we must be careful to use the $n - k$ -dimensional Fourier transform. Because of this we use the notation $\mathcal{F}_j$ to denote the Fourier transform on a j -dimensional vector subspace of $\mathbb{R}^n$. If $j = n$, then we use either $\mathcal{F}_n$ or $\mathcal{F}$.

Remark 3.42. The Riesz potential, which we will study later in section 3.7, is closely related to the lambda operator if $\alpha < 0$. In fact, in the case that $-n < \alpha < 0$ and u is rapidly decreasing the lambda operator is the same as the Riesz potential and is denoted by $I^{-\alpha}u$. (Compare, Rubin [524] for an extensive treatment of these ideas.) Also see Corollary 3.69.

In the case that α is an even positive integer and f is rapidly decreasing, the derivative theorem for the Fourier transform shows that $\Lambda^\alpha f$ is a constant multiple of a power of the Laplacian of f .

There is a large class of functions for which the lambda operator can be inverted. This class of functions includes the Schwartz class of rapidly decreasing functions. This inversion is the subject of the next result.

Proposition 3.43. *If f is a function in L^1 with the property that $|\xi|^{-\alpha} \widehat{f}(\xi) \in L^1$, then both $\Lambda^{-\alpha} f$ and $\Lambda^\alpha \Lambda^{-\alpha}(f)$ exist and we have the inversion formula*

$$\Lambda^\alpha \Lambda^{-\alpha}(f) = f.$$

Proof. The hypothesis that $|\xi|^{-\alpha} \widehat{f}(\xi) \in L^1$ shows that $\Lambda^{-\alpha}(f)$ is well defined because we can take the Fourier transform. If we define $u = \Lambda^{-\alpha}(f)$, then u is a tempered distribution, being the Fourier transform of a tempered distribution. Furthermore, $\widehat{u} = |\xi|^{-\alpha} \widehat{f}(\xi)$ by the definition of the lambda operator. Therefore, $|\xi|^\alpha \widehat{u}(\xi)$ is an essentially bounded function, because $f \in L^1$ implies $\widehat{f} \in L^\infty$. Therefore, $|\xi|^\alpha \widehat{u}(\xi)$ is a tempered distribution. Hence, $\Lambda^\alpha u$ is well defined and

$$\Lambda^\alpha u = \mathcal{F}^{-1}(|\xi|^\alpha \widehat{u}(\xi)) = \mathcal{F}^{-1}(|\xi|^\alpha |\xi|^{-\alpha} \widehat{f}(\xi)) = f.$$

But $\Lambda^\alpha u$ is also equal to $\Lambda^\alpha \Lambda^{-\alpha}(f)$ by definition. Combining these results proves the desired equation. ■

We now present an elementary inversion method for the k -plane transform. This inversion formula applies to certain integrable functions. In later sections we derive more general inversion formulas.

Theorem 3.44 (Inversion of the k -plane transform – I). *Let P be the k -plane transform, $0 < k < n$. If f is an L^1 function such that $|\xi|^{-\alpha} \widehat{f}(\xi) \in L^1$, then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^\alpha P^\# \Lambda^{k-\alpha} P f = f \quad (3.42)$$

Proof. The hypothesis that $|\xi|^{-\alpha} \widehat{f}(\xi)$ is integrable implies that $\Lambda^{-\alpha} f$ is well defined. Since $|\xi|^{-\alpha} \widehat{f}(\xi)$ is integrable, then its inverse Fourier transform, which is $\Lambda^{-\alpha} f(x)$, can be computed from an integral and, in fact, we have:

$$\Lambda^{-\alpha} f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |\xi|^{-\alpha} \widehat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi. \quad (3.43)$$

Because the integrand in this equation is integrable, we can use the generalized polar coordinate theorem to obtain three conclusions. First,

$$\int_{\mathbb{R}^n} |\xi|^{-\alpha} \widehat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi = \frac{1}{|G_{k,n-1}|} \int_{G_{k,n}} \int_{\eta^\perp} |\xi''|^{k-\alpha} \widehat{f}(\xi'') e^{i\langle x, \xi'' \rangle} d\xi'' d\eta, \quad (3.44)$$

second,

$$|\xi''|^{k-\alpha} \widehat{f}(\xi'') \in L^1(\eta^\perp) \text{ for almost all } \eta \in G_{k,n}, \quad (3.45)$$

and third, the function

$$\eta \mapsto \int_{\eta^\perp} |\xi''|^{k-\alpha} \widehat{f}(\xi'') e^{i\langle x, \xi'' \rangle} d\xi'' \text{ is integrable on } G_{k,n}. \quad (3.46)$$

We now examine the inner integral in equation 3.44 for hyperplanes η for which (3.45) is valid. The slice-projection theorem and the observation that if $\xi'' \in \eta^\perp$, then $\langle x, \xi'' \rangle = \langle \text{proj}_{\eta^\perp}(x), \xi'' \rangle$ transforms this integral to

$$\int_{\eta^\perp} |\xi''|^{k-\alpha} (2\pi)^{-\frac{k}{2}} \widehat{Pf}(\eta, \xi'') e^{i\langle \text{proj}_{\eta^\perp}(x), \xi'' \rangle} d\xi''$$

for which the integrand is L^1 on $\eta^\perp$. This is almost an inverse Fourier transform and indeed if we multiply by $(2\pi)^{k-\frac{n}{2}}$ we do have the following inverse Fourier transform on the $n - k$ -dimensional space $\eta^\perp$.

$$\begin{aligned} (2\pi)^{-\frac{n-k}{2}} \int_{\eta^\perp} |\xi''|^{k-\alpha} \widehat{Pf}(\eta, \xi'') e^{i\langle \text{proj}_{\eta^\perp}(x), \xi'' \rangle} d\xi'' \\ = \mathcal{F}_{n-k}^{-1} \left(|\xi''|^{k-\alpha} \widehat{Pf}(\eta, \xi'') \right) (\eta, \text{proj}_{\eta^\perp}(x)) \\ = \Lambda^{k-\alpha} Pf(\eta, \text{proj}_{\eta^\perp}(x)). \end{aligned}$$

We now use this result in equation (3.44). Remembering to multiply by $(2\pi)^{-(k-\frac{n}{2})}$ we get

$$\int_{\mathbb{R}^n} |\xi|^{-\alpha} \widehat{f}(\xi) e^{i\langle x, \xi \rangle} d\xi = (2\pi)^{-(k-\frac{n}{2})} \frac{1}{|G_{k,n-1}|} \int_{G_{k,n}} \Lambda^{k-\alpha} Pf(\eta, \text{proj}_{\eta^\perp}(x)) d\eta$$

Note that (3.46) implies that for each x the integral on the right-hand side of the previous equation exists and is finite for almost all η . Also note that this integral defines a backprojection. We now place this result in equation (3.43) which yields

$$\begin{aligned} \Lambda^{-\alpha} f(x) &= \frac{(2\pi)^{-k}}{|G_{k,n-1}|} \int_{G_{k,n}} \Lambda^{k-\alpha} Pf(\eta, \text{proj}_{\eta^\perp}(x)) d\eta \\ &= \frac{(2\pi)^{-k}}{|G_{k,n-1}|} P^\# \Lambda^{k-\alpha} Pf(x). \end{aligned} \quad (3.47)$$

Since f satisfies the hypotheses of proposition 3.43, we can conclude that $\Lambda^\alpha \Lambda^{-\alpha}(f) = f$. Therefore, applying Λ^α to equation (3.47) gives the desired result

$$f(x) = \frac{(2\pi)^{-k}}{|G_{k,n-1}|} \Lambda^\alpha P^\# \Lambda^{k-\alpha} Pf(x). \quad \blacksquare$$

Corollary 3.45. *Let P be the k -plane transform, $0 < k < n$. If p and α are real numbers with $1 \leq p \leq 2$ and $\frac{n}{p} < \alpha < n$ and if $f \in L^1 \cap L^p$, then we have the inversion formula*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^\alpha P^\# \Lambda^{k-\alpha} P f = f.$$

Proof. We only need to verify the hypotheses of the theorem, which are obvious except perhaps for $|\xi|^{-\alpha} \widehat{f}(\xi) \in L^1$. Let u_1 be the restriction of $|\xi|^{-\alpha} \widehat{f}(\xi)$ to the closed unit ball and let u_2 be the restriction to the complement. On the unit ball, the condition $\alpha < n$ implies that $|\xi|^{-\alpha} \in L^1$, whereas the condition $f \in L^1$ implies that $\widehat{f}(\xi)$ is bounded. Hence, the product $u_1 \in L^1$. On the complement of the unit ball $(|\xi|^{-\alpha})^p$ is integrable since $n < \alpha p$. Thus $|\xi|^{-\alpha} \in L^p$, whereas $\widehat{f} \in L^{p'}$ by the Hausdorff–Young theorem. Then by Hölder’s inequality the product, which is u_2 , is integrable. Hence, $|\xi|^{-\alpha} \widehat{f}(\xi) \in L^1$ and we see that all the hypotheses of the theorem have been verified. ■

Corollary 3.46. *If f is a compactly supported, square integrable function and $\frac{n}{2} < \alpha < n$, then the inversion formula (3.42) holds for f . In particular, if $\frac{n}{2} < k$, then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f = f.$$

Proof. Because any compactly supported L^2 function is integrable, then this follows directly from the previous corollary. ■

Remark 3.47. Inversion formulas of the type $\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f = f$ are particularly useful. Later on we will prove that we can remove the restriction $\frac{n}{2} < k$ in this corollary and indeed the result will be true for any k -plane transform and for any $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with $p \geq 2$; see corollary 3.81.

Another consequence of theorem 3.44 is an inversion formula for the k -plane transform of rapidly decreasing functions.

Corollary 3.48. *Let P be the k -plane transform, $0 < k < n$. If φ is a rapidly decreasing function and $\alpha < n$, then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^\alpha P^\# \Lambda^{k-\alpha} P \varphi = \varphi.$$

Proof. The condition $\alpha < n$ implies that $|\xi|^{-\alpha}$ is integrable near the origin. Any Schwartz class function is bounded, so $|\xi|^{-\alpha} \widehat{\varphi}(\xi)$ is integrable near the origin. Away from the origin, $|\xi|^{-\alpha}$ is bounded and Schwartz class functions are integrable. Putting this together shows that $|\xi|^{-\alpha} \widehat{\varphi}(\xi) \in L^1$. Since we already know that $S \subset L^1$, the hypotheses of the theorem are satisfied and we obtain the desired conclusion. ■

Remark 3.49. Theorem 3.44 in the case of the Radon transform is due to Smith, Solmon, and Wagner ([566]). Our proof is essentially the same as Natterer’s proof of theorem 2.1 in [444] except that we have generalized the result to arbitrary k -plane

transforms. Also, we have generalized the result from rapidly decreasing functions to the L^1 functions described in theorem 3.44.

3.6.1 Inversion Formulas for Smooth Functions

If we restrict attention to functions with infinitely many derivatives, then the inversion formulas can be presented in even more explicit form. For convenience we state the results for the Schwartz class $\mathcal{S}$ of rapidly decreasing functions.

The Laplacian differential operator on $\mathbb{R}^n$ is defined by $\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$. There is an inversion theorem based on the Laplacian.

Theorem 3.50. *If $f \in \mathcal{S}$ and k is even, then $f = \frac{(-1)^{\frac{k}{2}}}{(2\pi)^k |G_{k,n-1}|} \Delta^{\frac{k}{2}} P^\# P f$.*

Proof. From the derivative theorem for the Fourier transform and the definition of Λ^α we get

$$\begin{aligned} \widehat{\Delta f}(\xi) &= \sum (i\xi_j)^2 \widehat{f}(\xi) \\ &= -|\xi|^2 \widehat{f}(\xi) = -(\widehat{\Lambda^2 f}) \end{aligned}$$

and from this it is immediate that

$$\Delta^j f = (-1)^j \Lambda^{2j} f.$$

Now if we use the inversion formula in theorem 3.44 for the k -plane transform on the Schwartz class with k even and $\alpha = k$ we get

$$\begin{aligned} f &= \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f \\ &= \frac{(-1)^{\frac{k}{2}}}{(2\pi)^k |G_{k,n-1}|} \Delta^{\frac{k}{2}} P^\# P f \end{aligned}$$

■

Applying this result to the Radon transform on odd dimensional spaces gives:

Corollary 3.51. *If $f \in \mathcal{S}$ and n is odd, then $f = \frac{(-1)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}} \Delta^{\frac{n-1}{2}} R^\# R f$*

In particular, if $n = 3$, and f is rapidly decreasing on $\mathbb{R}^3$, then

$$f(x) = -\frac{1}{8\pi^2} \Delta R^\# R f \quad (3.48)$$

which is a result obtained by Radon in his 1917 paper [508] (compare, chapter 1 and introduction of Helgason [291])

Proof. If n is odd and we set $k = n - 1$, then k is even and $P^\# = \frac{1}{2} R^\#$ (theorem 3.28). From the theorem we then have

$$f = \frac{(-1)^{\frac{n-1}{2}}}{(2\pi)^{n-1} |G_{n-1,n-1}|} \Delta^{\frac{n-1}{2}} \left(\frac{1}{2} R^\# \right) R f$$

which gives the result since $|G_{n-1,n-1}| = 1$.

If $n = 3$ then we get $f(x) = -\frac{1}{8\pi^2} \Delta R^\# Rf$. ■

We require the Hilbert transform for the next step.

Definition 3.52. The **Hilbert transform** Hf of a function f defined on the real line is the principal value integral $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$. More precisely,

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

It is known that if $f \in L^p(\mathbb{R})$ for $1 < p < \infty$, then Hf exists almost everywhere and is in fact a bounded linear operator on $L^p(\mathbb{R})$ (chapter 6, theorem 2.6 of Stein and Weiss [583]). Also, the Hilbert transform of a rapidly decreasing function exists and has a Fourier transform given by

$$\widehat{Hf}(t) = -i \operatorname{sgn}(t) \widehat{f}(t)$$

(chapter 7.1 of Natterer [444]; chapter 1, lemma 3.7 of Helgason [291]).

If f is a function of two variables, then we take the Hilbert transform with respect to the second variable.

Note that $\widehat{H^2 f}(t) = -i \operatorname{sgn}(t) \widehat{Hf}(t) = (-i \operatorname{sgn}(t))(-i \operatorname{sgn}(t) \widehat{f}(t)) = (-i \operatorname{sgn}(t))^2 \widehat{f}(t)$. Iterating this relation gives

$$\widehat{H^k f}(t) = (-i \operatorname{sgn}(t))^k \widehat{f}(t). \quad (3.49)$$

Recall that if P denotes the $n-1$ plane transform and if R denotes the Radon transform, then by equation (3.35)

$$Rf(\theta, p) = Pf(\theta^\perp, p\theta).$$

Derivatives of the Radon transform with respect to the second variable will be denoted by $\frac{\partial Rf}{\partial p}$. It can be shown that if $f \in \mathcal{S}$, then $Rf(\theta, p)$ is in the Schwartz class in the second variable (Helgason [291]). Therefore we can always apply the Hilbert transform, in the second variable, to derivatives of the Radon transform.

The next result gives explicit inversion formulas for the Radon transform in terms of backprojections of derivatives of the Radon transform. Special cases of this result were obtained by Radon [508], John [322], Helgason [269], Gel'fand, Graev, and Vilenkin [194], and Ludwig [405]. The particular derivation given here, based on theorem 3.44, appears in Natterer [444].

Theorem 3.53. Let $f \in \mathcal{S}$. Then

1. If n is odd, then

$$f(x) = \frac{(-1)^{\frac{n-1}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

and

2. If n is even, then

$$f(x) = \frac{(-1)^{\frac{n-2}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} H \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

Proof. Note that $|t| = \operatorname{sgn}(t)t$ so

$$|t|^{n-1} = (\operatorname{sgn}(t)t)^{n-1} = (-i \operatorname{sgn}(t))^{n-1} (it)^{n-1}.$$

From this we get

$$\mathcal{F}(\Lambda^{n-1} Rf)(\theta, t) = (-i \operatorname{sgn}(t))^{n-1} (it)^{n-1} \widehat{Rf}(\theta, t).$$

The derivative theorem for the Fourier transform gives $(it)^{n-1} \widehat{Rf}(\theta, t) = \mathcal{F}(\frac{\partial^{n-1} Rf}{\partial p^{n-1}})(\theta, t)$ and, hence,

$$\begin{aligned} \mathcal{F}(\Lambda^{n-1} Rf)(\theta, t) &= (-i \operatorname{sgn}(t))^{n-1} \mathcal{F}\left(\frac{\partial^{n-1} Rf}{\partial p^{n-1}}\right)(\theta, t) \\ &= \mathcal{F}\left(H^{n-1} \frac{\partial^{n-1} Rf}{\partial p^{n-1}}\right)(\theta, t) \end{aligned} \quad (3.50)$$

The last step is by equation (3.49). Also, by equation (3.49) we easily obtain $H^2 Rf = -Rf$. This leads to: $H^{n-1} \frac{\partial^{n-1} Rf}{\partial p^{n-1}} = (-1)^{\frac{n-1}{2}} \frac{\partial^{n-1} Rf}{\partial p^{n-1}}$ if n is odd, whereas $H^{n-1} \frac{\partial^{n-1} Rf}{\partial p^{n-1}} = (-1)^{\frac{n-2}{2}} H \frac{\partial^{n-1} Rf}{\partial p^{n-1}}$ if n is even. Putting these results into the previous equation and canceling the Fourier transform results in

$$\Lambda^{n-1} Rf(\theta, p) = \begin{cases} (-1)^{\frac{n-1}{2}} \frac{\partial^{n-1} Rf}{\partial p^{n-1}} & \text{if } n \text{ is odd} \\ (-1)^{\frac{n-2}{2}} H \frac{\partial^{n-1} Rf}{\partial p^{n-1}} & \text{if } n \text{ is even} \end{cases} \quad (3.51)$$

Since $f \in \mathcal{S}$, we can use corollary 3.48 with $\alpha = 0$ and $k = n - 1$ to get $f = \frac{1}{(2\pi)^{n-1}} P^\# \Lambda^{n-1} P f$. Using theorem 3.28 we can write this as $f = \frac{1}{(2\pi)^{n-1}} (\frac{1}{2} R^\#) \Lambda^{n-1} R f$. Substituting equation (3.51) into this expression proves the theorem. ■

3.7 Riesz Potentials and the Backprojection Theorem

The backprojection theorem is an important relation that shows that the backprojection of the k -plane transform, $P^\# P$, is a convolution operator. More precisely we will show that there exists a function R_k such that

$$P^\# P f = c \cdot R_k * f \quad (3.52)$$

where c is a constant depending only on k and n . The factor R_k is called a Riesz kernel and the convolution itself is an example of a Riesz potential. These ideas will be made precise in definition 3.55. Equation 3.52 was originally derived by Fuglede [183] and we will present the proof of his theorem shortly. The backprojection theorem, which we present as theorem 3.67 below, is a somewhat more general version of Fuglede's theorem.

If we could invert the Riesz potential operator, then, using equation 3.52 we could recover f purely from the knowledge of its k -plane transform.

The Riesz potential can be inverted on large classes of functions. For example one has the results of Rubin [524, 525, 528, 538] and Samko, Kilbas, and Marichev [542] (compare, Smith and Keinert [564] and Keinert [348]). Using these general results on the inversion of the Riesz potential we could state and prove inversion results for the k -plane transform that are much more general than theorem 3.44 or corollary 3.48. These proofs would be very short; however, the background needed would violate the prerequisites that we have established here. Therefore, we will develop an inversion formula based on a multiplier theorem for the Riesz potential. The idea of using a multiplier theorem originates with Smith and Keinert [564] and Keinert [348]. However their proof also required a substantial overhead in prerequisites. We are able to develop this approach using only very elementary properties of the Riesz potential, which we derive with full details, along with some elementary facts about real analysis and distributions.

We now prove Fuglede's theorem.

Theorem 3.54 (Fuglede [183]). *Let k be an integer with $0 < k < n$. If f is a non-negative Lebesgue measurable function on $\mathbb{R}^n$, then we have $P^\# P f(x) = |G_{k-1, n-1}| \int_{\mathbb{R}^n} |w|^{k-n} f(x-w) dw$.*

Proof. In the following discussion x is a fixed point in $\mathbb{R}^n$ and therefore acts as a constant in all integrations. The nonnegativity of the function f will ensure the validity of any changes of variable, although the integrals involved may be infinite.

From the definitions of the k -plane transform and its formal adjoint we immediately have

$$\begin{aligned} P^\# P f(x) &= \int_{G_{k,n}} P f(\eta, \text{proj}_{\eta^\perp}(x)) d\eta \\ &= \int_{G_{k,n}} \int_{\eta} f(y' + \text{proj}_{\eta^\perp}(x)) dy' d\eta \end{aligned}$$

Now fix $\eta \in G_{k,n}$ and consider the inner integral in the previous equation. Make the substitution $w' = \text{proj}_{\eta}(x) - y'$ in the inner integral. Because both y' and $\text{proj}_{\eta}(x)$ are in the linear space η , we see that this is a valid substitution. Also since $x = \text{proj}_{\eta}(x) + \text{proj}_{\eta^\perp}(x)$ we see that $x - w' = \text{proj}_{\eta^\perp}(x) + y'$ so

$$\begin{aligned} P^\# P f(x) &= \int_{G_{k,n}} \int_{\eta} f(x - w') dw' d\eta \\ &= \int_{G_{k,n}} \int_{\eta} |w'|^{n-k} (|w'|^{k-n} f(x - w')) dw' d\eta \end{aligned}$$

Since f is nonnegative, as is $|w|^{k-n}$, we can apply the generalized polar coordinate theorem to the function $h(w) = |w|^{k-n} f(x - w)$, which, in conjunction with the previous

equation, shows that

$$P^\# P f(x) = |G_{k-1, n-1}| \int_{\mathbb{R}^n} |w|^{k-n} f(x-w) dw$$

■

3.7.1 Definition and Elementary Properties of Riesz Potentials

In this section α will denote a positive real number.

We will define the Riesz potential to be a constant multiple $\rho_{\alpha, n}$ of the convolution integral

$$\int_{\mathbb{R}^n} |w|^{\alpha-n} f(x-w) dw \quad (3.53)$$

which occurred in Fuglede's formula in the case $\alpha = k$. It is customary to make a somewhat unusual choice for the constant $\rho_{\alpha, n}$:

$$\rho_{\alpha, n} = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}. \quad (3.54)$$

We leave it to the reader to verify that in the case that $\alpha = k$ is an integer then we can also express this coefficient as

$$\rho_{k, n} = (2\pi)^{-k} \frac{|S^{k-1}|}{|S^{n-k-1}|} \quad (3.55)$$

We then define the **Riesz kernel** of order α to be the function

$$R_\alpha(x) = \rho_{\alpha, n} |x|^{\alpha-n} \quad (3.56)$$

Definition 3.55. If f is a function defined on $\mathbb{R}^n$ and if $0 < \alpha < n$, then the **Riesz potential** I^α of order α is the transform defined by

$$I^\alpha(f) = R_\alpha * f$$

where $*$ denotes convolution.

Hence,

$$I^\alpha f(x) = \rho_{\alpha, n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

The kernels R_α and the convolution operators $R_\alpha * f$ were introduced by M. Riesz [521].

Remark 3.56. The reason for the particular choice of the constant $\rho_{\alpha, n}$ is that with this choice the following equation for the Fourier transform of the Riesz potential is valid for a large class of functions

$$\widehat{R_\alpha * f}(\xi) = |\xi|^{-\alpha} \widehat{f}(\xi)$$

For example, this is true when f is rapidly decreasing (compare, theorem 3.68 and Stein [581]).

Remark 3.57. It is a historical accident that the Radon transform and the Riesz potential are closely connected and share the same symbol R . However, most of our uses of the Riesz potential occur in connection with the k -plane transform P , and if the issue should ever arise any ambiguity will be resolved by the context.

Remark 3.58. There is a close connection between Riesz potentials and the lambda operator (definition 3.39). We will amplify this point as we learn more about Riesz potentials.

The Riesz kernel is a nonnegative measurable function on $\mathbb{R}^n$, but it is never integrable. However, its product with certain functions may be integrable, so there is some hope that the Riesz potentials $I^\alpha(f)$ will be finite almost everywhere if we are careful about choosing the function f . We will consider this question soon. Although R_α is not integrable, it is a tempered function since $\frac{R_\alpha(x)}{(1+|x|^2)^\alpha}$ is an L^1 function. Hence:

Proposition 3.59. *The Riesz kernel R_α is a tempered distribution.*

The Riesz potential $R_\alpha * f$ behaves most nicely on functions f such that $f(x)(1+|x|)^{\alpha-n}$ is integrable, as illustrated in the next few results. The main results are that $R_\alpha * f$ is finite almost everywhere if (and in a sense only if) $f(x)(1+|x|)^{\alpha-n}$ is integrable and, in this case, $R_\alpha * f$ is a tempered distribution.

Lemma 3.60. *Let $0 < \alpha < n$. There is a constant C such that for any measurable function ϕ and any $t \in \mathbb{R}^n$ we have*

$$|R_\alpha * \phi(t)| \leq C(1+|t|)^{\alpha-n} (\|\phi\|_{L^1} + \|(1+|x|)^n \phi(x)\|_{L^\infty}). \quad (3.57)$$

The constant depends only on n, α .

The proof of this lemma is elementary, albeit tedious. It can be accomplished by decomposing the integral defining the convolution $R_\alpha * \phi(t)$ and estimating each resulting piece. For completeness, we have provided the details in the appendix (see section 3.14.2).

Lemma 3.61. *If $(1+|x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$, if $\phi \in L^1(\mathbb{R}^n)$ and if $(1+|x|)^n \phi(x) \in L^\infty(\mathbb{R}^n)$, then the function*

$$F(x, y) = \rho_{\alpha, n} |x - y|^{\alpha-n} f(y) \overline{\phi(x)}$$

is integrable on $\mathbb{R}^n \times \mathbb{R}^n$. Furthermore we have the estimate

$$\int \int_{\mathbb{R}^n \times \mathbb{R}^n} |F(x, y)| dx dy \leq C \|(1+|x|)^{\alpha-n} f(x)\|_{L^1} (\|\phi\|_{L^1} + \|(1+|x|)^n \phi(x)\|_{L^\infty}),$$

where C is a constant depending only on α and n .

Proof. Integrate $F(x, y)$ with respect to x keeping y fixed. Then we obtain the following estimate in which C is the constant from lemma 3.60:

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x, y)| dx &\leq |f(y)| \int_{\mathbb{R}^n} \rho_{\alpha, n} |x - y|^{\alpha-n} |\phi(x)| dx \\ &= |f(y)| (R_\alpha * |\phi|)(y) \\ &\leq C |f(y)| (1 + |y|)^{\alpha-n} (\|\phi\|_{L^1} + \|(1 + |x|)^n \phi(x)\|_{L^\infty}) \end{aligned}$$

Since $(1 + |y|)^{\alpha-n} f(y) \in L^1(\mathbb{R}^n)$, we can integrate the previous inequality to obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |F(x, y)| dx \right) dy \\ \leq C \|(1 + |y|)^{\alpha-n} f(y)\|_{L^1} (\|\phi\|_{L^1} + \|(1 + |x|)^n \phi(x)\|_{L^\infty}) \\ < \infty. \end{aligned}$$

By Fubini's theorem it follows that F is integrable on $\mathbb{R}^n \times \mathbb{R}^n$ and from the previous inequality the estimate in the conclusion is true. ■

Theorem 3.62. *Let $0 < \alpha < n$. Then there is a constant C , depending only on α and n , such that for any measurable functions ϕ and f we have*

$$\begin{aligned} |\langle R_\alpha * f, \phi \rangle| \\ \leq C \|(1 + |x|)^{\alpha-n} f(x)\|_{L^1} (\|\phi\|_{L^1} + \|(1 + |x|)^n \phi(x)\|_{L^\infty}). \end{aligned} \quad (3.58)$$

Note that if $f \in L^1(\mathbb{R}^n, (1 + |x|)^{\alpha-n})$ and $\phi \in \mathcal{S}$, then the right-hand side of this inequality is finite.

Proof. There is no loss in generality by assuming that ϕ is an integrable function such that $(1 + |x|)^n \phi(x)$ is essentially bounded on $\mathbb{R}^n$ and that $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$, since if not then the right-hand side of inequality (3.58) is infinite and therefore the inequality is true by default. Therefore, we can apply lemma 3.61. Because the function f in that lemma is integrable on $\mathbb{R}^n \times \mathbb{R}^n$ we can reverse the order of integration to get

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x, y) dy \right) dx &= \int_{\mathbb{R}^n} \left(\rho_{\alpha, n} \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y) dy \right) \overline{\phi(x)} dx \\ &= \int_{\mathbb{R}^n} (R_\alpha * f)(x) \overline{\phi(x)} dx = \langle R_\alpha * f, \phi \rangle \end{aligned}$$

This combined with the estimate in the lemma yields

$$|\langle R_\alpha * f, \phi \rangle| \leq C \|(1 + |x|)^{\alpha-n} f(x)\|_{L^1} (\|\phi\|_{L^1} + \|(1 + |x|)^n \phi(x)\|_{L^\infty}).$$

■

Theorem 3.63. *If $0 < \alpha < n$ and if $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$, then $R_\alpha * f$ is a tempered distribution.*

Proof. In this proof φ denotes an arbitrary function in the Schwartz class $\mathcal{S}$. The integral $C_1 = \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} dx$ is finite, so

$$\begin{aligned} \|\varphi\|_{L^1} &= \int_{\mathbb{R}^n} |\varphi(x)| dx = \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} |\varphi(x)| (1+|x|)^{n+1} dx \\ &\leq C_1 \left\| (1+|x|)^{n+1} \varphi(x) \right\|_{L^\infty} \end{aligned}$$

The constant $C_2 = \left\| (1+|x|)^{\alpha-n} f(x) \right\|_{L^1}$ is finite by hypothesis. From this and theorem 3.62 we find that there is a constant C such that

$$|\langle R_\alpha * f, \varphi \rangle| \leq C C_2 \left(C_1 \left\| (1+|x|)^{n+1} \varphi(x) \right\|_{L^\infty} + \left\| (1+|x|^n \varphi(x) \right\|_{L^\infty} \right)$$

and this is enough to prove that $R_\alpha * f$ is a tempered distribution. ■

Theorem 3.64. *Let $0 < \alpha < n$. If $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$, then $R_\alpha * f$ is finite almost everywhere and is a locally integrable function.*

Proof. We can use theorem 3.62 with the given f and with ϕ the characteristic function of the ball centered at 0 of radius r to conclude that $\int_{|x| \leq r} |(R_\alpha * f)(x)| dx = \langle R_\alpha * |f|, \phi \rangle$ is finite. This shows that $R_\alpha * f$ is locally integrable. Also this implies that $(R_\alpha * f)(x)$ exists as an absolutely convergent integral at almost any point in the ball $B(r)$. Since $\mathbb{R}^n$ is covered by countably many such balls we find that $(R_\alpha * f)(x)$ exists for almost all $x \in \mathbb{R}^n$. ■

In counterpoint to these results we have the following dramatic nonexistence property.

Theorem 3.65. *If $0 < \alpha < n$, $f \geq 0$ and $(1 + |x|)^{\alpha-n} f(x) \notin L^1(\mathbb{R}^n)$, then $R_\alpha * f = \infty$ almost everywhere.*

Proof. The hypotheses that $(1 + |x|)^{\alpha-n} f(x) \notin L^1(\mathbb{R}^n)$ and $f \geq 0$ imply that $\int_{\mathbb{R}^n} (1 + |y|)^{\alpha-n} f(y) dy = \infty$. Now we consider two cases. First if $|x| \leq 1$, then $|x - y| \leq 1 + |y|$ so

$$R_\alpha * f(x) = \rho_{\alpha,n} \int_{\mathbb{R}^n} |x - y|^{\alpha-n} f(y) dy \geq \rho_{\alpha,n} \int_{\mathbb{R}^n} (1 + |y|)^{\alpha-n} f(y) dy = \infty.$$

In the second case, where $|x| > 1$, we see that $|x - y| \leq |x| + |y| \leq |x| + |x||y| = |x|(1 + |y|)$ and a similar calculation shows $R_\alpha * f(x) = \infty$. ■

Theorems 3.64 and 3.65 immediately lead to the following characterization of non-negative functions whose Riesz potentials are finite almost everywhere.

Theorem 3.66. *A nonnegative measurable function f defined on $\mathbb{R}^n$ has the property that $R_\alpha * f$ is finite almost everywhere if and only if $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$.*

3.7.2 The Backprojection Theorem

Having developed the elementary properties of the Riesz potential, we can now generalize Fuglede's theorem, thus arriving at the backprojection theorem.

Theorem 3.67 (The backprojection theorem). *If P is the k -plane transform, $0 < k < n$, and f is a nonnegative Lebesgue measurable function on $\mathbb{R}^n$, then we have*

$$P^\# P f = (2\pi)^k |G_{k,n-1}| R_k * f \quad (3.59)$$

Furthermore, if $f \in L^1(\mathbb{R}^n)$, or even more generally, if $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, then equation (3.59) is not only true almost everywhere, but is finite almost everywhere on both sides.

Proof. Let us start with f being a nonnegative Lebesgue measurable function on $\mathbb{R}^n$. From Fuglede's theorem we have

$$\begin{aligned} P^\# P f(x) &= |G_{k-1,n-1}| \int_{\mathbb{R}^n} |w|^{k-n} f(x-w) dw \\ &= \frac{|G_{k-1,n-1}|}{\rho_{k,n}} \int_{\mathbb{R}^n} (\rho_{k,n} |w|^{k-n}) f(x-w) dw \\ &= \frac{|G_{k-1,n-1}|}{\rho_{k,n}} (R_k * f). \end{aligned} \quad (3.60)$$

It is an easy exercise using the definition of the Grassmannian volume, equation (3.18) together with equation (3.55) to show that $\frac{|G_{k-1,n-1}|}{\rho_{k,n}} = (2\pi)^k |G_{k,n-1}|$. This proves equation (3.59) in the case that f is a nonnegative measurable function.

Note that if $f \in L^1(\mathbb{R}^n)$, then the condition $k < n$ implies $(1 + |x|)^{k-n} f(x) \leq f(x)$, and by dominated convergence we see that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. Therefore, it suffices to prove the finiteness part of the theorem for the case where $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$.

There is no loss in generality by assuming that f is real valued. Let f^+ be the positive part of f : $f^+(x) = f(x)$, if $f(x) \geq 0$, otherwise $f^+(x) = 0$. Also let f^- , the negative part of f , be defined by $f^-(x) = -f(x)$, if $f(x) < 0$, otherwise $f^-(x) = 0$. Then $f = f^+ - f^-$ and both f^+ and f^- are nonnegative measurable functions. Furthermore, the positive part of $(1 + |x|)^{k-n} f(x)$ is clearly equal to $(1 + |x|)^{k-n} f^+(x)$ with a similar result for the negative part. Since $(1 + |x|)^{k-n} f(x)$ is integrable, then the same is true of its positive and negative parts.

Because both $(1 + |x|)^{k-n} f^+(x)$ and $(1 + |x|)^{k-n} f^-(x)$ are in L^1 , then by theorem 3.64 for almost all x it is true that the integrals defining $R_k * f^+(x)$ and $R_k * f^-(x)$ are finite Lebesgue integrals. Therefore, the integral defining $R_k * f(x)$ exists as a finite Lebesgue integral for almost all x . This combined with equation (3.60) shows that $P^\# P f(x) = (2\pi)^k |G_{k,n-1}| R_k * f(x)$ is true and finite on both sides for almost all x . ■

With the backprojection theorem we are now able to provide a proof of theorem 3.36, which states:

Theorem. *A nonnegative measurable function f defined on $\mathbb{R}^n$ has the property that Pf exists almost everywhere and is locally integrable on $\mathcal{G}_{k,n}$ if and only if $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$.*

Proof. Sufficiency of the condition $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ is established by theorem 3.35. For necessity, assume that f is a nonnegative measurable function f defined on $\mathbb{R}^n$ such that Pf exists almost everywhere and is locally integrable on $\mathcal{G}_{k,n}$. Let χ_r denote the characteristic function of the ball of radius r centered at the origin. Let us consider the inner product $\langle P\chi_r, Pf \rangle$. Since χ_r is a radial function vanishing for $|x| > r$, theorem 3.31 shows that $P\chi_r(\eta, x'') = 0$ for $|x''| > r$. Therefore, the inner product is defined by an integral on the compact measure space formed by taking all $(\eta, x'') \in \mathcal{G}_{k,n}$ with $|x''| \leq r$. Because Pf is assumed to be locally integrable and $P\chi_r$ is clearly bounded, then we see that the integral defining $\langle P\chi_r, Pf \rangle$ is finite. Now we use the formal adjoint theorem to get $\infty > \langle P\chi_r, Pf \rangle = \langle \chi_r, P^\#Pf \rangle$. Applying the backprojection theorem yields $\infty > \langle \chi_r, (2\pi)^k |G_{k,n-1}| R_k * f \rangle$, which implies that $\int_{|x| \leq r} (R_k * f)(x) dx < \infty$. Since f is nonnegative, theorem 3.65 shows that we must have $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. ■

3.8 An Inversion Formula for the k -Plane Transform of L^2 Functions

In this section we derive a fairly general inversion theorem for the k -plane transform that applies to L^2 functions satisfying a certain condition on the Fourier transform. The inversion formula is

$$f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# Pf, \quad (3.61)$$

which we already know is true for rapidly decreasing functions and certain L^1 functions.

The derivation of the inversion formula in this section depends on the Fourier transform properties of Riesz potentials. If one has a background in fractional integrals equivalent to that of [524] by Rubin, then significantly more general results can be obtained. For example, Rubin [525, 528 and 538] develops analogous inversion formulas for certain L^p functions. However, we have taken the current approach to be consistent with the prerequisites that we prescribed for this chapter. We do describe Rubin's results in section 3.10. Also, there is another approach to inversion formulas for the k -plane transform that is based on an extension of the k -plane transform as an unbounded Hilbert space operator. This approach may be found in section 3.9.

The backprojection theorem shows that $P^\#Pf$ is a multiple of the Riesz potential $R_k * f$. If we could invert the Riesz potential, then we would obtain an inversion formula for the k -plane transform similar to equation (3.61).

Fourier inversion would be helpful for inverting the Riesz potential, but the kernel R_k is neither L^1 nor L^2 . However, it is a tempered distribution and thus has a Fourier transform as a distribution. Let us try to determine its Fourier transform in a very informal way. This will lead to a conjecture for the Fourier transform of the Riesz

potential. The kernel R_k is homogeneous of degree $k - n$ so its Fourier transform is homogeneous of degree $-k$. Therefore we expect the Fourier transform of R_k to be a constant multiple of $|\xi|^{-k}$. Indeed, this turns out to be true and

$$\widehat{R_k}(\xi) = (2\pi)^{-\frac{n}{2}} |\xi|^{-k} \quad (3.62)$$

where this is meant in the sense of tempered distributions: for every φ in the Schwartz class $\mathcal{S}$ we have $\langle \widehat{R_k}, \varphi \rangle = \langle (2\pi)^{-\frac{n}{2}} |\xi|^{-k}, \varphi \rangle$ (Stein [581] and Helgason [291] (especially lemma 5.2)).² From the convolution theorem for the Fourier transform we might expect to have $\widehat{R_k * f} = (2\pi)^{\frac{n}{2}} \widehat{R_k} \widehat{f}$, which combined with (3.62) would give $\widehat{R_k * f} = |\xi|^{-k} \widehat{f}$.

However, the relation $\widehat{R_k * f} = |\xi|^{-k} \widehat{f}$ does not follow from the convolution theorem for the Fourier transform because R_k is not an integrable function. In fact this relation is not true even for some relatively nice functions (see Smith and Solmon [565], page 544, for an L^2 function for which this relation fails).

Nonetheless we will be able to prove that $\widehat{R_k * f} = |\xi|^{-k} \widehat{f}$ for a large class of functions f . Once we have this relation we can invert the Riesz potential via the Λ operator. Recalling our earlier discussion of multiplier theorems, it is convenient to call the relation $\widehat{R_k * f} = |\xi|^{-k} \widehat{f}$ a **Riesz multiplier theorem**. We begin by proving the Riesz multiplier theorem for rapidly decreasing functions (theorem 3.68). We can do this for integer values of k purely by the theory of the k -plane transform, without proving that $\widehat{R_k}(\xi) = (2\pi)^{-\frac{n}{2}} |\xi|^{-k}$. The Riesz multiplier theorem for rapidly decreasing functions is then used later to prove a much more general Riesz multiplier theorem, theorem 3.74, which is one of the main results of this section, because a fairly general inversion theorem for the k -plane transform follows almost immediately from this theorem.

The original formulation of the Riesz multiplier theorem 3.74 was by Smith and Solmon [565] who proved it under the conditions that f be an L^2 function, that $(1 + |x|)^{k-n} f(x)$ be integrable, and that either side of the equation $\widehat{R_k * f}(\xi) = |\xi|^{-k} \widehat{f}(\xi)$ be square integrable. A more general version, still in the L^2 case, was proved by Smith and Keinert [564] (for $k = 1$, the x-ray transform) and by Keinert [348] (for general values of k). Using this result, Smith and Keinert were able to invert the x-ray transform and Keinert was able to invert the k -plane transform.

Throughout this section we assume, unless otherwise stated, that k and n are given integers with $0 < k < n$, and that P denotes the k -plane transform on $\mathbb{R}^n$.

3.8.1 A Multiplier Theorem for the Riesz Potential

We begin by proving the Riesz multiplier theorem for rapidly decreasing functions. This result is well known. Stein [581] has a proof that depends on some results on about singular integrals, whereas Helgason [291] has a fairly long proof depending on

² Because there is no standard normalization of the Fourier transform, equation (3.62) appears with different constants in certain references. For example, Stein [581] defines the Fourier transform as $\int f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ and equation (3.62) appears there in the form $\widehat{R_k}(\xi) = (2\pi)^{-k} |\xi|^{-k}$, whereas Helgason [291] defines the Fourier transform as $\int f(x) e^{-i \langle x, \xi \rangle} dx$ and equation (3.62) appears there as $\widehat{R_k}(\xi) = |\xi|^{-k}$. Our equation (3.62) is correct given our normalization of the Fourier transform.

Also, in equation (3.62) k may be any real number, $0 < k < n$.

the theory of distributions.³ Our proof is brief and depends only on some of the theory of the k -plane transform that we have already developed. However, our proof is only valid for integers k with $0 < k < n$, whereas the other proofs are valid for any real number in that range. But our result suffices to obtain the desired inversion theorem for the k -plane transform.

Theorem 3.68 (Riesz multiplier theorem for the Schwartz class). *If φ is a function in the Schwartz class $\mathcal{S}$, and if k is an integer with $0 < k < n$, then*

$$\widehat{R_k * \varphi}(\xi) = |\xi|^{-k} \widehat{\varphi}(\xi). \quad (3.63)$$

Proof. By theorem 3.48 we have

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P \varphi = \varphi$$

and by the backprojection theorem we have

$$P^\# P \varphi = (2\pi)^k |G_{k,n-1}| R_k * \varphi.$$

Combining these results gives $\Lambda^k (R_k * \varphi) = \varphi$ for any rapidly decreasing function φ . Hence, we get

$$\varphi(\xi) = \mathcal{F}^{-1} \left(|\xi|^k \widehat{R_k * \varphi}(\xi) \right)$$

Taking the Fourier transform, which is possible because φ is rapidly decreasing, yields

$$\widehat{\varphi}(\xi) = |\xi|^k \widehat{R_k * \varphi}(\xi)$$

and the theorem follows directly from this equation. ■

The next result illustrates the connection between the Riesz potential and the lambda operator that we mentioned earlier.

Corollary 3.69. *If $\varphi \in \mathcal{S}$, then $R_k * \varphi = \Lambda^{-k} \varphi$.*

Proof. Apply the inverse Fourier transform to equation (3.63). ■

We now extend the Riesz multiplier theorem $(\widehat{R_k * f})(\xi) = |\xi|^{-k} \widehat{f}(\xi)$ to certain L^2 functions. This result appears in theorem 3.74 after we prove the following preliminary results. Some of the proofs are deferred to the appendix (see Section 3.14).

Lemma 3.70. *Suppose that $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$. Then for any $\varphi \in \mathcal{S}$ we have*

$$\langle R_\alpha * f, \varphi \rangle = \langle f, R_\alpha * \varphi \rangle$$

and the result is always finite.

³ For the reader who is not familiar with meromorphic extensions of distributions, the discussion in sections 2–4 of Jensen [321] is a helpful supplement to [291]. Jensen's discussion parallels Helgason's but is somewhat more elementary and has more details.

Proof. In the following computation there is a change in the order of integration that is justified by Fubini's theorem and lemma 3.61:

$$\begin{aligned}
 \langle R_\alpha * f, \varphi \rangle &= \int_{\mathbb{R}^n} (R_\alpha * f)(x) \overline{\varphi(x)} dx \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \rho_{\alpha,n} |x - y|^{\alpha-n} f(y) \overline{\varphi(x)} dy dx \\
 &= \int_{\mathbb{R}^n} f(y) \overline{\int_{\mathbb{R}^n} \rho_{\alpha,n} |x - y|^{\alpha-n} \varphi(x) dx} dy \\
 &= \langle f, R_\alpha * \varphi \rangle.
 \end{aligned}$$

■

Lemma 3.71. *If $\varphi \in \mathcal{S}$ and $\partial^\beta \varphi(0) = 0$ for all multi-indices with $|\beta| < k$, then there are functions φ_α , also in the Schwartz class $\mathcal{S}$, such that for any $\xi \in \mathbb{R}^n$ we have*

$$\varphi(\xi) = \sum_{|\alpha|=k} \xi^\alpha \varphi_\alpha(\xi)$$

Proof. Deferred to the appendix (section 3.14). ■

Lemma 3.72. *Let $0 < k < n$ and let $e \in L_0^\infty$. Define $e_r(x) = r^{-n} e\left(\frac{x}{r}\right)$ for $r > 0$. Then*

1. *If $\widehat{u} \in L^1 + L^2$, then $\langle u, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$.*
2. *If $(1 + |x|)^{k-n} f(x) \in L^1$, then $\langle R_k * f, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$.*

Proof. Deferred to the appendix (section 3.14). ■

Lemma 3.73. *Let $f \in L^2(\mathbb{R}^n)$ be such that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. Then for any rapidly decreasing function with the property that $\partial^\alpha \varphi(0) = 0$ for all multi-indices with $|\alpha| < k$ we have*

$$\left\langle \widehat{R_k * f}(\xi) - |\xi|^{-k} \widehat{f}(\xi), \varphi(\xi) \right\rangle = 0. \quad (3.64)$$

Proof. By lemma 3.71 there are functions φ_α , also in the Schwartz class $\mathcal{S}$, such that $\varphi(\xi) = \sum_{|\alpha|=k} \xi^\alpha \varphi_\alpha(\xi)$. Recalling that $\widetilde{\varphi}$ denotes the inverse Fourier transform of φ , we now prove that

$$R_k * \widetilde{\varphi} = \mathcal{F}^{-1} \left(|\xi|^{-k} \varphi(\xi) \right) \quad \text{and} \quad (3.65)$$

$$R_k * \widetilde{\varphi} \in L^2(\mathbb{R}^n). \quad (3.66)$$

Theorem 3.68 shows that $\widehat{R_k * \widetilde{\varphi}} = |\xi|^{-k} \varphi(\xi)$ and this establishes (3.65). Because the inverse Fourier transform of an L^2 function is again L^2 , equation (3.65) shows that to prove (3.66) it suffices to prove that $|\xi|^{-k} \varphi(\xi) \in L^2$. For this it suffices to prove that each $|\xi|^{-k} \xi^\alpha \varphi_\alpha(\xi) \in L^2$. Note that $|\xi^\alpha| = |\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n} \leq |\xi|^k$, since $|\alpha| = k$. Therefore $|\xi|^{-k} |\xi^\alpha| \leq 1$ and this shows that $\| |\xi|^{-k} \xi^\alpha \varphi_\alpha(\xi) \|_{L^2} \leq \|\varphi_\alpha\|_{L^2} < \infty$, since $\varphi_\alpha \in \mathcal{S}$. This establishes (3.66).

In the next step we begin by using the Plancherel property of the Fourier transform of tempered distributions [equation (3.7)] and then we use lemma 3.70 to commute R_k in the resulting inner product, thus obtaining

$$\begin{aligned} \langle \widehat{R_k * f}, \varphi \rangle &= \langle \widehat{R_k * f}, \widehat{\tilde{\varphi}} \rangle \\ &= \langle R_k * f, \tilde{\varphi} \rangle = \langle f, R_k * \tilde{\varphi} \rangle \end{aligned} \quad (3.67)$$

By the hypothesis and (3.66), $\langle f, R_k * \tilde{\varphi} \rangle$ is an inner product of L^2 functions. With Plancherel's theorem this is the same as $\langle \widehat{f}, \widehat{R_k * \tilde{\varphi}} \rangle$, which equals $\langle \widehat{f}, |\xi|^{-k} \varphi(\xi) \rangle$ by equation (3.65). If we now move the factor $|\xi|^{-k}$ to the first term of the inner product we then obtain $\langle \widehat{R_k * f}, \varphi \rangle = \langle |\xi|^{-k} \widehat{f}, \varphi(\xi) \rangle$, which establishes the lemma. ■

We now derive the main result of this section. It is a multiplier theorem for Riesz potentials that is from Smith and Keinert [564], for $k = 1$, and Keinert [348] for $0 < k < n$.

Before proceeding to the theorem let us review some facts about distributions. If u is a tempered distribution with compact support, then u extends to a compactly supported distribution, that is, for any C^∞ function ψ , we can define $u(\psi)$ in such a way as to make u a continuous linear functional on the space $\mathcal{E} = C^\infty(\mathbb{R}^n)$. This is done by letting ϕ_0 be a rapidly decreasing function that equals 1 on a neighborhood of $\text{supp}(u)$. Then the product $\phi_0 \psi$ is easily seen to be rapidly decreasing. Therefore $u(\phi_0 \psi)$ is defined and we can take this to be the value of $u(\psi)$ when we extend u from $\mathcal{S}$ to $\mathcal{E}$.

It can be shown that the inverse Fourier transform of a compactly supported distribution u is a function and is defined by $\tilde{u}(\xi) = u(\exp(ix \cdot \xi))$, where, for each fixed x , the argument of the linear functional u is the C^∞ function of ξ defined by the given exponential.

An example of a compactly supported distribution is the Dirac δ distribution. It is defined by $\delta(\varphi) = \varphi(0)$ and its support is the origin. By the definition of distributional derivatives we have

$$(\partial^\alpha \delta)(\varphi) = (-1)^{|\alpha|} (\partial^\alpha \varphi)(0). \quad (3.68)$$

We now compute the inverse Fourier transform of $\partial^\alpha \delta$. We do this by the method outlined above: $\tilde{\partial^\alpha \delta}(x)$ is the value of the distribution $\partial^\alpha \delta$ applied to the C^∞ function $\exp(ix \cdot \xi)$, where in this function x is fixed and ξ is the variable. This gives $\tilde{\partial^\alpha \delta}(x) = (\partial^\alpha \delta)(\exp(ix \cdot \xi))$. By equation (3.68), the last expression is the same as the function $(-1)^{|\alpha|} \partial_\xi^\alpha (\exp(ix \cdot \xi))$ evaluated at $\xi = 0$. The subscript in ∂_ξ^α indicates differentiation with respect to ξ , with x fixed. A short calculation shows that this results in the expression $(-1)^{|\alpha|} (ix)^\alpha \exp(ix \cdot \xi)$. Evaluating this at $\xi = 0$ and combining with the other computations results in the formula

$$\tilde{\partial^\alpha \delta}(x) = i^{-|\alpha|} x^\alpha. \quad (3.69)$$

We now prove the Riesz multiplier theorem for Riesz potentials of order k , where k is any real number with $0 < k < n$. The proof depends on the Riesz multiplier theorem for rapidly decreasing functions which we proved for integer k in theorem 3.68. But as

we remarked there, the theorem in the case of rapidly decreasing functions is known for any real number k with $0 < k < n$.

Theorem 3.74 (Riesz multiplier theorem). *Let k be a real number with $0 < k < n$. Let $f \in L^2(\mathbb{R}^n)$ be such that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ and $|\xi|^{-k} \widehat{f}(\xi) \in L^1_{\text{loc}}$. Then*

$$(\widehat{R_k * f})(\xi) = |\xi|^{-k} \widehat{f}(\xi) \quad (3.70)$$

and we have the following inversion formula for the Riesz potential:

$$\Lambda^k(R_k * f) = f. \quad (3.71)$$

Proof. Let $u = \widehat{R_k * f} - |\xi|^{-k} \widehat{f}$. By lemma 3.73 u is a tempered distribution that vanishes when applied to any function ϕ in the Schwartz class S such that $\partial^\alpha \phi(0) = 0$ for all multi-indices with $|\alpha| < k$. In particular, any compactly supported C^∞ function ϕ that vanishes in a neighborhood of the origin satisfies this condition. Therefore, u is a compactly supported distribution and its support consists of the single point $\{0\}$.

If γ is any function in the Schwartz class, then we can use Taylor's theorem to express γ as follows:

$$\gamma(x) = \sum_{|\alpha| < k} \frac{\partial^\alpha \gamma(0)}{\alpha!} x^\alpha + R(x), \quad (3.72)$$

where the remainder term $R(x)$ is a C^∞ function such that $\partial^\alpha R(0) = 0$ for all multi-indices with $|\alpha| < k$. This relation will also be true for $\phi_0 R$, where ϕ_0 is any rapidly decreasing function which equals 1 on a neighborhood of the origin. Then $\phi_0 R$ is rapidly decreasing function and hence $u(R) = u(\phi_0 R)$, since u is compactly supported. Furthermore the properties of R and ϕ_0 and the Leibniz product formula for derivatives imply that $\partial^\alpha(\phi_0 R)(0) = 0$ for $|\alpha| < k$. Hence, $u(R) = 0$ by lemma 3.73. From this and an application of u to equation (3.72) we get

$$u(\gamma) = \sum_{|\alpha| < k} \frac{\partial^\alpha \gamma(0)}{\alpha!} u(x^\alpha) = \sum_{|\alpha| < k} \frac{(-1)^{|\alpha|} u(x^\alpha)}{\alpha!} (\partial^\alpha \delta)(\gamma),$$

where the last step follows from equation (3.68). Also note that $u(x^\alpha)$ is well defined because u is compactly supported and x^α is a C^∞ function. Since γ is an arbitrary function in S , then $u = \sum_{|\alpha| < k} \frac{(-1)^{|\alpha|} u(x^\alpha)}{\alpha!} \partial^\alpha \delta$ and the definition of u shows that

$$\widehat{R_k * f} = |\xi|^{-k} \widehat{f}(\xi) + \sum_{|\alpha| < k} \frac{(-1)^{|\alpha|} u(x^\alpha)}{\alpha!} \partial^\alpha \delta$$

Letting w be the inverse Fourier transform of $|\xi|^{-k} \widehat{f}(\xi)$, taking the inverse Fourier

transform in the previous equation and using equation (3.69) gives

$$\begin{aligned} R_k * f &= w + \sum_{|\alpha| < k} \frac{(-1)^{|\alpha|} u(x^\alpha)}{\alpha!} i^{-|\alpha|} x^\alpha \\ &= w + \sum_{|\alpha| < k} c_\alpha x^\alpha \\ &= w + q, \end{aligned} \tag{3.73}$$

where $c_\alpha = \frac{i^{|\alpha|} u(x^\alpha)}{\alpha!}$ and $q(x) = \sum_{|\alpha| < k} c_\alpha x^\alpha$.

We now show that the polynomial q is identically zero. To do this start by fixing $(t_1, \dots, t_n) \in \mathbb{R}^n$ and letting e be the characteristic function of the rectangle

$$[0, t_1] \times \dots \times [0, t_n].$$

The definition of w and the hypothesis that $|\xi|^{-k} \widehat{f}(\xi)$ is integrable near the origin shows that $\widehat{w}$ is integrable near the origin. We now decompose $\widehat{w}$ into two pieces. The first piece is obtained by restricting $\widehat{w}$ to a ball around the origin. This piece is therefore L^1 . The second piece is L^2 , since $\widehat{f}$ is L^2 and $|\xi|^{-k}$ is bounded away from the origin. This shows that $\widehat{w} \in L^1 + L^2$.

We now use lemma 3.72 in which we defined $e_r(x) = r^{-n} e\left(\frac{x}{r}\right)$. In our situation e_r is r^{-n} times the characteristic function of the rectangle $[0, rt_1] \times \dots \times [0, rt_n]$. By the first part of lemma 3.72 we get $\lim_{r \rightarrow \infty} \langle w, e_r \rangle = 0$. By the second part of that lemma and the hypothesis on f we also get $\lim_{r \rightarrow \infty} \langle R_k * f, e_r \rangle = 0$. Using this in equation (3.73) gives $\langle q, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$. Using the fact that e_r is r^{-n} times the characteristic function of the rectangle $[0, rt_1] \times \dots \times [0, rt_n]$, we then get

$$\langle q, e_r \rangle = \sum_{|\alpha| < k} c_\alpha \int_0^{rt_1} \dots \int_0^{rt_n} x^\alpha r^{-n} dx = \sum_{|\alpha| < k} c_\alpha \frac{t_1^{\alpha_1+1}}{\alpha_1+1} \dots \frac{t_n^{\alpha_n+1}}{\alpha_n+1} r^{|\alpha|}.$$

Since $(t_1, \dots, t_n) \in \mathbb{R}^n$ is fixed, this last expression is a polynomial in the single variable r . The only way that its limit can vanish at infinity is if the polynomial is identically zero.

This means that for fixed $(t_1, \dots, t_n) \in \mathbb{R}^n$ we have $\sum_{|\alpha| < k} c_\alpha \frac{t_1^{\alpha_1+1}}{\alpha_1+1} \dots \frac{t_n^{\alpha_n+1}}{\alpha_n+1} = 0$. But this is true for all $(t_1, \dots, t_n) \in \mathbb{R}^n$ so the polynomial $\sum_{|\alpha| < k} c_\alpha \frac{t_1^{\alpha_1+1}}{\alpha_1+1} \dots \frac{t_n^{\alpha_n+1}}{\alpha_n+1}$ in n variables is identically zero. Therefore, all its coefficients are zero and we obtain $c_\alpha = 0$ for $|\alpha| < k$. This implies that q is identically 0 and by equation (3.73) $R_k * f = w$. Because w is the inverse Fourier transform of $|\xi|^{-k} \widehat{f}(\xi)$, taking the Fourier transform in the previous equation gives conclusion (3.70).

From equation (3.70) we get $|\xi|^k \widehat{R_k * f}(\xi) = \widehat{f}(\xi)$, which is in L^2 because f is square integrable. In particular, $|\xi|^k \widehat{R_k * f}$ is a tempered distribution. Therefore, $R_k * f$ is in the domain of Λ^k and we thus have

$$\Lambda^k (R_k * f) = \mathcal{F}^{-1} \left(|\xi|^k \widehat{(R_k * f)}(\xi) \right) = \mathcal{F}^{-1} (\widehat{f}) = f.$$

This completes the proof. ■

Remark 3.75. We remarked earlier that this theorem originated with Smith and Keinert ([564]) and Keinert ([348]). Although our proof contains elements of the proof developed by Smith and Keinert, the approach here is much more elementary because we do not rely on the Calderón–Zygmund theory of singular integrals and some other fairly deep areas of analysis used by Smith and Keinert.

Corollary 3.76. *If $0 < \alpha < n$, $2 \leq p < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then*

$$(\widehat{R_\alpha * f})(\xi) = |\xi|^{-\alpha} \widehat{f}(\xi)$$

and we have the following inversion formula for the Riesz potential:

$$\Lambda^k(R_k * f) = f.$$

Compactly supported L^p functions, $2 \leq p < \infty$, and Schwartz class functions are particular examples of functions of this type.

Proof. First we prove that $f \in L^2$. To do this let $A = \{x \in \mathbb{R}^n : |f(x)| > 1\}$ and let B be the complement of A . Then $f = f_A + f_B$, where f_A and f_B are the restrictions of f to the sets A and B , respectively. Since $p \geq 2$ and $|f(x)| > 1$ on the set A , we have $|f(x)|^2 \leq |f(x)|^p$ on A . Since f , and hence f_A , is L^p , it follows that $\int_{\mathbb{R}^n} |f_A(x)|^2 dx < \infty$ and hence f_A is L^2 .

On the set B we have $|f(x)| \leq 1$ so $\int_B |f(x)|^2 \leq \int_{\mathbb{R}^n} |f(x)| dx < \infty$ since $f \in L^1$. Thus f_B is also L^2 .

Hence, $f \in L^2(\mathbb{R}^n)$.

Since $\alpha < n$ and $f \in L^1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} (1 + |x|)^{\alpha-n} |f(x)| dx \leq \int_{\mathbb{R}^n} |f(x)| dx < \infty.$$

The hypothesis that f is L^1 also implies, by the Riemann–Lebesgue lemma, that $\widehat{f}$ is a bounded continuous function, so $\|\widehat{f}\|_{L^\infty}$ is finite.

If we now choose $r > 0$ we then get

$$\int_{B(r)} |\xi|^{-\alpha} |\widehat{f}(\xi)| d\xi \leq \|\widehat{f}\|_{L^\infty} \int_{B(r)} |\xi|^{-\alpha} d\xi < \infty$$

because $\alpha < n$. This shows that $|\xi|^{-\alpha} \widehat{f}(\xi)$ is locally integrable. All the hypotheses of theorem 3.74 are satisfied and the corollary is proved. ■

Corollary 3.77. *If $0 < \alpha < n$, if f is a measurable function such that $f(x) = O(|x|^{-N})$ for some $N > n$ and which is either L^2 or bounded near the origin, then $(\widehat{R_\alpha * f})(\xi) = |\xi|^{-\alpha} \widehat{f}(\xi)$ and we have the following inversion formula for the Riesz potential:*

$$\Lambda^k(R_k * f) = f.$$

Proof. There is no loss in generality by assuming that f is bounded on the ball $B(1)$.

The assumption $f(x) = O(|x|^{-N})$ means that there is a constant C such that $|f(x)||x|^N < C$ for all $x \in \mathbb{R}^n$. Hence, $\int_{|x|>1} |f(x)|dx$ is bounded by a constant multiple of $\int_{|x|>1} |x|^{-N}dx$, which is finite since $N > n$. This together with the fact that f is bounded on $B(1)$ shows that $f \in L^1(\mathbb{R}^n)$. In the same way we can show that $f \in L^2(\mathbb{R}^n)$ and the result for bounded functions follows from the preceding corollary. The proof for L^2 functions is similar and is omitted. ■

Remark 3.78. Helgason [291] has a different version of this corollary. His statement uses the following notation: I^k denotes the Riesz potential if $k \geq 0$ and denotes Λ^k if $k < 0$. Helgason's version reads $I^{-k}I^k f = f$ if f is a C^∞ function, which is $O(|x|^{-N})$ for some $N > n$. In a recent result, Jensen [321] proved that $I^{-k}I^k f = f$ if f is $O(|x|^{-N})$ for some $N > k$ and f satisfies a weak Hölder condition.

All the functions routinely used in tomography, including functions in the Schwartz class $\mathcal{S}$ and compactly supported L^2 functions are in $L^1 \cap L^2$ and hence satisfy the multiplier theorem for the Riesz potential.

3.8.2 The Inversion Theorem

The following inversion theorem originated with Smith and Keinert ([564]) for $k = 1$ and Keinert ([348]) for general k .

Theorem 3.79 (Inversion of the k -plane transform–II). *Let $f \in L^2(\mathbb{R}^n)$ be such that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ and $|\xi|^{-k} \widehat{f}(\xi) \in L^1_{\text{loc}}$, then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f = f. \quad (3.74)$$

Proof. The hypotheses allow the use of the Riesz multiplier theorem to get $\Lambda^k(R_k * f) = f$. Furthermore, since $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, we can use the backprojection theorem to get $R_k * f(x) = \frac{1}{(2\pi)^k |G_{k,n-1}|} P^\# P f(x)$, for almost all x . Combining these equations gives the desired result. ■

Corollary 3.80 (Inversion of the Radon Transform). *Let $f \in L^2(\mathbb{R}^n)$ be such that $(1 + |x|)^{-1} f(x) \in L^1(\mathbb{R}^n)$ and $|\xi|^{-n+1} \widehat{f}(\xi) \in L^1_{\text{loc}}$, then*

$$\frac{1}{2^n \pi^{n-1}} \Lambda R^\# R f = f.$$

Proof. Applying the theorem with $k = n - 1$ yields

$$f = \frac{1}{(2\pi)^{n-1} |G_{n-1,n-1}|} \Lambda^{n-1} P^\# P f. \quad (3.75)$$

We know that $|G_{n-1,n-1}| = 1$ and by theorem 3.28 we have, except for unimportant distinctions in the domains, $P^\# = \frac{1}{2} R^\#$ and $P = R$. Substituting these values in equation (3.75) gives the desired result. ■

Corollaries 3.76 and 3.77 yield the following important special cases of the inversion theorem.⁴

Corollary 3.81. *If $0 < k < n$, $2 \leq p < \infty$ and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then*

$$f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f.$$

In particular, this inversion theorem is true for compactly supported L^p functions, $2 \leq p < \infty$, and Schwartz class functions.

Corollary 3.82. *If $0 < k < n$, if f is a measurable function such that $f(x) = O(|x|^{-N})$ for some $N > n$ and which is either L^2 or bounded near the origin, then*

$$f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f. \quad (3.76)$$

Remark 3.83. Helgason [291] has a similar inversion formula for continuous functions which are $O(|x|^{-N})$ for $N > n$ (theorems 3.1 and 6.2 in [291]). Recently, Jensen [321] proved the much more general result that the inversion formula (3.76) is valid if f is $O(|x|^{-N})$ for some $N > k$ and f satisfies a weak Hölder condition (compare, Rubin [538] and also corollary 3.124).

Remark 3.84. Versions of this inversion formula have been known for quite a while. The case $k = n - 1$ can be traced back to Radon's 1917 paper [508]. The inversion formula for the k -plane transform first appeared in a 1959 paper by Helgason [265]. His result applied to functions satisfying the $O(|x|^{-N})$ condition of the corollary. We note that Fuglede [183] in 1958 already had, in our notation, the formula $P^\# P f = c \cdot R_k * f$, but he did not invert the Riesz potential to obtain an inversion of the k -plane transform. Other early contributors to the inversion of the k -plane transform are Gel'fand, Graev, and Vilenkin [194], Solmon [569], and Smith, Solmon, and Wagner [566]. Most of these results assumed a fair amount of smoothness for the function f . The usual assumption was for f to be in the Schwartz class of rapidly decreasing functions or for f to be continuous and $O(|x|^{-N})$, where $N > n$, although Solmon [569] had a more general result (see section 3.9). Rubin has several inversion formulas of this nature for the k -plane transform of L^p functions. We present more details on Rubin's results in section 3.10.

Remark 3.85. Some authors use different normalizations of the formal adjoint $P^\#$. For example, Helgason [291] and Jensen [321] have inversion theorems of the form

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} I^{-k}(\hat{f})^\vee,$$

⁴ The condition $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ is necessary to compute the k -plane transform. However, this condition is satisfied in both corollaries 3.81 and 3.82 as shown in the proofs of corollaries 3.76 and 3.77.

where I is the Riesz potential operator, $\widehat{f}$ is the k -plane transform of f , and $g^\vee$ denotes Helgason's form of the formal adjoint. One would think that in our notation this would be $f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n}{2})} \Lambda^k P^\# P f$. But then the constant would be wrong. Actually $P^\# g = |G_{k,n}| g^\vee$, and when this is taken into account, both inversion formulas are the same. In essence, Helgason has taken the volume of the Grassmannian to be 1, whereas we have defined it differently. On the other hand, our definition of the formal adjoint is more explicit.

In the case that $k < \frac{n}{2}$, the inversion theorem does not require $(1 + |x|)^{k-n} f(x) \in L^1$ and $|\xi|^{-k} \widehat{f}(\xi) \in L^1_{\text{loc}}$.

Corollary 3.86. *If $0 < k < \frac{n}{2}$, then for any square integrable function f we have the inversion formula*

$$f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f.$$

Proof. Since $k < \frac{n}{2}$, then $2(k - n) < -n$, which implies that $(1 + |x|)^{k-n} \in L^2$. Since $f \in L^2$ also, the Cauchy-Schwarz inequality implies that the product $(1 + |x|)^{k-n} f(x) \in L^1$.

Furthermore, $2k < n$, which implies that $|\xi|^{-k}$ is L^2 on any ball centered at the origin. Since $\widehat{f}$ is L^2 , the same reasoning that we already used shows that the product $|\xi|^{-k} \widehat{f}(\xi)$ is integrable on any ball and hence is locally integrable. This shows that the hypotheses, and hence the conclusion, of theorem 3.79 are valid. ■

These corollaries show that there are many examples of functions for which the inversion theorem for the k -plane transform holds. In all these examples we were able to derive the fact that $|\xi|^{-k} \widehat{f}(\xi)$ is integrable. In the last example, the inversion formula applied to all L^2 functions with $k < \frac{n}{2}$. It is natural to ask whether the hypotheses $(1 + |x|)^{k-n} f(x) \in L^1$ and $|\xi|^{-k} \widehat{f}(\xi) \in L^1_{\text{loc}}$ are necessary if $k \geq \frac{n}{2}$.

Certainly, the hypothesis $(1 + |x|)^{k-n} f(x) \in L^1$ is necessary in this case because theorems 3.32 and 3.36 show that there are L^2 functions such that $(1 + |x|)^{k-n} f(x) \notin L^1$ and such that the k -plane transform of f does not exist.

The next question then is whether the conditions $f \in L^2$ and $(1 + |x|)^{k-n} f(x) \in L^1$ imply $|\xi|^{-k} \widehat{f}(\xi) \in L^1_{\text{loc}}$. E. M. Stein (private communication) has provided an example to show that the answer to this question is no. Here is a sketch of his construction. Start with an L^2 function f such that $|\xi|^{-k} \widehat{f}(\xi)$ is not integrable in a neighborhood of the origin. This is easy to do. Simply cook up a nonnegative function F which is L^2 and for which $|\xi|^{-k} F(\xi)$ has an infinite integral on the unit ball. If we define f to be the inverse Fourier transform of F , then $f \in L^2$ and $|\xi|^{-k} \widehat{f}(\xi) \notin L^1_{\text{loc}}$. However, there is no guarantee that $(1 + |x|)^{k-n} f(x) \in L^1$. So we define g_j to be the restriction of f to the ball of radius j centered at the origin. It is well known that each such $g_j \in L^1 \cap L^2$ and that the sequence g_j converges to f in L^2 -norm. We now translate each g_j . Define $f_j(x) = g_j(x - d_j)$, where d_j is a constant to be determined. Note that if d_j is large enough, then we can make $\|(1 + |x|)^{k-n} f_j(x)\|_{L^1}$ as small as we please. This follows

because $k - n$ is negative, so

$$\begin{aligned} \int_{\mathbb{R}^n} (1 + |x|)^{k-n} |f_j(x)| dx &\leq d_j^{k-n} \|f_j\|_{L^1} \\ &= d_j^{k-n} \|g_j\|_{L^1} \end{aligned}$$

(the L^1 norms of f_j and g_j are the same because they are translates of each other). Starting with g_1 we then inductively define d_j and f_j so that the integrals $\int_{\mathbb{R}^n} (1 + |x|)^{k-n} |f_j(x)| dx$ are sufficiently small and so that the supports of the f_j are disjoint. One needs to take subsequences at this stage, but let us ignore this subtlety. It is then fairly routine to show that if we judiciously choose the constants d_j and then define $f^* = \sum_{j=1}^{\infty} j^{-j} f_j$, then f^* turns out to be square integrable and f^* satisfies $(1 + |x|)^{k-n} f^*(x) \in L^1$. Since each f_j is a translate of g_j and since $\widehat{g_j} \rightarrow \widehat{f}$ in L^2 – norm, then it is relatively simple, modulo some annoying calculations, to show that $|\xi|^{-k} \widehat{f}(\xi) \notin L^1_{\text{loc}}$ leads to $|\xi|^{-k} f^*(\xi) \notin L^1_{\text{loc}}$. The function f^* then provides the desired example.

3.9 The k -Plane Transform as an Unbounded Operator on L^2

In this section we show how the k -plane transform can be extended from the Schwartz class to a fairly large class of L^2 functions. The extended k -plane transform is denoted by $\overline{P}$ and its domain by $D_{\overline{P}}$. There are functions in $D_{\overline{P}}$ for which the k -plane integrals defining the usual k -plane transform do not exist. Therefore, the extended k -plane transform of such a function is defined by a limiting process in Hilbert space, not by actual computations of integrals. Nevertheless, we show that much of the usual theory persists: for example, there is a slice-projection theorem and an inversion formula for the extended k -plane transform.

The inversion formula for this extended k -plane transform takes the form

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^{k-s} P^\# \Lambda^s \overline{P} f = f.$$

The allowable values of the parameter s depend on the smoothness of the function f .

In the case $s = 0$ this inversion formula takes the same form as our previous inversion formula in theorem 3.79. But the class of functions for which our new approach works is different than the class for theorem 3.79. This point is pursued following corollary 3.96.

We start by defining some spaces of L^2 functions related to the lambda operator.

Definition 3.87. Define subspaces of L^2 by

$$\begin{aligned} \mathcal{D}_s(\mathbb{R}^n) &= \left\{ f \in L^2(\mathbb{R}^n) : |\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n) \right\} \\ \mathcal{D}_s(\mathcal{G}_{k,n}) &= \left\{ g \in L^2(\mathcal{G}_{k,n}) : |\xi''|^s \widehat{g}(\eta, \xi'') \in L^2(\mathcal{G}_{k,n}) \right\}. \end{aligned}$$

As usual, the Fourier transform is taken with respect to the second variable on $\mathcal{G}_{k,n}$. In general, we use the symbol $\mathcal{D}_s$ for both $\mathcal{D}_s(\mathbb{R}^n)$ and $\mathcal{D}_s(\mathcal{G}_{k,n})$. The confusion is minimal and is always resolved by the context. For $s \geq 0$, $\mathcal{D}_s$ is the usual Sobolev space $H^{\frac{s}{2}}(\mathbb{R}^n)$, but this is not true for $s < 0$.

We can define the spaces $\mathcal{D}_s$ for any real number, but, for reasons that will appear later, we assume throughout this section that s is a real number satisfying $s > -\frac{n}{2}$.

In informal terms, the space $\mathcal{D}_s$ consists of all square integrable functions f for which $\Lambda^s f$ is defined and is square integrable. In fact we have:

Proposition 3.88. $f \in \mathcal{D}_s$ if and only if both f and $\Lambda^s f$ are L^2 functions.

Proof. It suffices to show that $|\xi|^s \widehat{f}(\xi) \in L^2$ if and only if $\Lambda^s f \in L^2$. But this follows directly from the definition of the lambda operator and Plancherel's theorem. ■

Many common functions are in these spaces.

Theorem 3.89. If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and if $-\frac{n}{2} < s \leq 0$, then $f \in \mathcal{D}_s(\mathbb{R}^n)$.

If $s > -\frac{n}{2}$, then the Schwartz class $\mathcal{S}$ of rapidly decreasing functions is contained in $\mathcal{D}_s$.

Proof. First suppose that $f \in L^1 \cap L^2$ and $-\frac{n}{2} < s \leq 0$. In this case it suffices to show that $|\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$. Since f is L^1 , then by Riemann–Lebesgue, $\widehat{f}(\xi)$ is bounded. Also $|\xi|^s$ is square integrable near the origin since $2s > -n$. Hence $|\xi|^s \widehat{f}(\xi)$ is L^2 near the origin. Away from the origin, $|\xi|^s$ is bounded since $s \leq 0$ and therefore $|\xi|^s \widehat{f}(\xi)$ is L^2 away from the origin since $f \in L^2 \implies \widehat{f} \in L^2$.

The statement about the Schwartz class is proved in a similar manner. ■

We now proceed to show that we can extend the k -plane transform from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{D}_s(\mathbb{R}^n)$. The next two lemmas describe a few properties that we require of the $\mathcal{D}_s$ spaces.

Lemma 3.90. $\mathcal{S}$ is dense in $\mathcal{D}_s$. Moreover, given any $f \in \mathcal{D}_s$ and any $\varepsilon > 0$, we can find $\varphi \in \mathcal{S}$ such that

$$\|f - \varphi\|_{L^2} < \varepsilon$$

and

$$\| |\xi|^s \widehat{f}(\xi) - |\xi|^s \widehat{\varphi}(\xi) \|_{L^2} < \varepsilon.$$

Proof. Let $f \in \mathcal{D}_s(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given.

Case I. $s \geq 0$. By definition $|\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, so some $R = R_\varepsilon > 0$ must exist such that $\int_{|\xi| \geq R_\varepsilon} (|\xi|^s |\widehat{f}(\xi)|)^2 d\xi < \frac{\varepsilon^2}{2}$. There is no loss in generality by assuming $R > 1$, in which case we have $|\xi|^s \geq 1$ on the set where $R > 1$. Let A be the set where $|\xi| \leq R$ and let A' denote its complement. Because C_0^∞ is dense in L^2 , we can choose some Φ which is C^∞ and supported on A such that $\|\widehat{f}(\xi) - \Phi\|_{L^2(A)} < \frac{\varepsilon}{\sqrt{2}R^s}$. Hence,

$$\begin{aligned} \|\widehat{f}(\xi) - \Phi\|_{L^2(\mathbb{R}^n)}^2 &= \|\widehat{f}(\xi) - \Phi\|_{L^2(A)}^2 + \|\widehat{f}(\xi) - \Phi\|_{L^2(A')}^2 \\ &= \|\widehat{f}(\xi) - \Phi\|_{L^2(A)}^2 + \|\widehat{f}(\xi)\|_{L^2(A')}^2 \end{aligned}$$

The last step is valid since $\Phi = 0$ on A' . We already have an estimate for the first term on the right. The second term is no greater than $\int_{|\xi| \geq R_\varepsilon} ||\xi|^s \widehat{f}(\xi)|^2 d\xi$, since $s \geq 0$ and $|\xi| > 1$ on A' . But we also had an estimate for this integral. Putting all this together shows that

$$\|\widehat{f}(\xi) - \Phi\|_{L^2}^2 \leq \frac{\varepsilon^2}{2R^{2s}} + \frac{\varepsilon^2}{2} < \varepsilon^2.$$

If we now define φ to be the inverse Fourier transform of Φ , then $\varphi \in \mathcal{S}$ and by the Plancherel theorem we have

$$\|f - \varphi\|_{L^2(\mathbb{R}^n)}^2 < \varepsilon^2.$$

A similar calculation gives the estimate $\||\xi|^s \widehat{f}(\xi) - |\xi|^s \Phi(\xi)\|_{L^2}^2 < R^{2s} \frac{\varepsilon^2}{2R^{2s}} + \frac{\varepsilon^2}{2} = \varepsilon^2$ and this completes the proof if $s \geq 0$.

The second case where $s < 0$ is handled in much the same way. ■

A direct consequence of this lemma is:

Lemma 3.91. *For any $f \in \mathcal{D}_s$, there exists a sequence φ_j of rapidly decreasing functions in $\mathcal{D}_s(\mathbb{R}^n)$ such that $\varphi_j \rightarrow f$ in L^2 norm and $|\xi|^s \widehat{\varphi}_j(\xi) \rightarrow |\xi|^s \widehat{f}(\xi)$ in L^2 norm.*

Note that $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{D}_s(\mathbb{R}^n)$ are vector subspaces of the Hilbert space $L^2(\mathbb{R}^n)$. However, if $s > 0$, then $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{D}_s(\mathbb{R}^n)$ are not closed subspaces of $L^2(\mathbb{R}^n)$ and, therefore, are not Hilbert spaces with the usual L^2 inner product on $\mathbb{R}^n$. On the other hand, $\mathcal{D}_s(\mathbb{R}^n)$ being a Sobolev space when $s > 0$ is a Hilbert space, but only with the Sobolev norm.

The k -plane transform restricted to $L^2(\mathbb{R}^n)$ is an unbounded linear operator. This is also true of the k -plane transform restricted to $\mathcal{S}(\mathbb{R}^n)$. We illustrate this for the Radon transform. We start by showing that the Radon transform is unbounded on $L^2_0(\mathbb{R}^n)$. Consider the sequence of functions f_j , where each f_j is the characteristic function of the ball of radius j . By proposition 2.66 of chapter 2 we know that $Rf_j(\theta, t) = c \cdot (j^2 - t^2)^{\frac{n-1}{2}}$. If we set up the integral defining the L^2 norm on $S^{n-1} \times \mathbb{R}$ and make the change of variable $w^2 = j^2 - t^2$, then we get $\|Rf_j\|_{L^2}^2 = c \int_0^j w^{2n-1} \frac{1}{\sqrt{j^2 - w^2}} dw \geq \frac{c}{j} \int_0^j w^{2n-1} dw = c \cdot j^{2n-1}$. But $\|f_j\|_{L^2}^2$ is just the volume of the ball of radius j and hence is of the order j^n . Therefore there cannot be any constant M such that $\|Rf_j\|_{L^2} \leq M\|f_j\|_{L^2}$. By smoothing the characteristic functions we can extrapolate this result to show that the Radon transform is an unbounded operator on $\mathcal{S}(\mathbb{R}^n)$.

Nonetheless, we can extend the k -plane transform P as an unbounded operator from rapidly decreasing functions to a larger subspace of L^2 . We denote the larger subspace of L^2 by $D_{\overline{P}}$ and the extension of the k -plane transform to this subspace by $\overline{P}$.

Definition 3.92. Let $D_{\overline{P}}$ be the set of functions f with the property that there exists a sequence φ_j of rapidly decreasing functions such that φ_j converges to f in $L^2(\mathbb{R}^n)$ and there exists a function $g \in L^2(\mathcal{G}_{k,n})$ such that $P\varphi_j$ converges to g in $L^2(\mathcal{G}_{k,n})$. We

then define $\overline{P}f = g$, we call $\overline{P}$ the **extended k -plane transform**, and we call $D_{\overline{P}}$ the **domain of the extended k -plane transform**.

In the case $k = n - 1$ we have the **extended Radon transform** that we denote by $\overline{R}$. As usual, we take $(\theta, p) \in S^{n-1} \times \mathbb{R}$ to be the arguments of $\overline{R}f$ rather than $(\eta, x'') \in \mathcal{G}_{n-1,n}$.

Proposition 3.93. *The operator $\overline{P}$ from definition 3.92 is well defined and is a linear operator with domain $D_{\overline{P}}$ and range contained in $L^2(\mathcal{G}_{k,n})$.*

The proof is straightforward and is omitted except for the comment that one can easily show that different sequences in the definition lead to the same value for $\overline{P}f$.

The slice-projection theorem states that for L^1 functions f we have $\overline{P}f(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')$, where $\eta \in G_{k,n}$ and $\xi'' \in \eta^\perp$. We intend to extend this theorem to functions in $D_{\overline{P}}$.

Theorem 3.94 (Slice projection for the extended k -plane transform). *If $f \in D_{\overline{P}}$ then for almost every k plane η and for almost every $\xi'' \in \eta^\perp$ we have*

$$(\overline{P}f)^\wedge(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'') \quad (3.77)$$

Proof. We use the fact that if X is a measure space with measure μ , and if (f_n) is a sequence of functions in $L^p(X, d\mu)$, which converges to a function $f \in L^p(X, d\mu)$, then there is some subsequence of (f_n) , which converges to f almost everywhere with respect to the measure μ . In this case we say that there is an “appropriate subsequence” with this property and we keep the same notation (f_n) for the subsequence.

By definition 3.92 there exists a sequence φ_j of rapidly decreasing functions and a function $g \in L^2(\mathcal{G}_{k,n})$ such that $\varphi_j \rightarrow f$ and $P\varphi_j \rightarrow g$, the convergence being L^2 in the appropriate measure space. Then $\overline{P}f = g$.

In particular, we have $\lim_{j \rightarrow \infty} \|g - P\varphi_j\|_{L^2(\mathcal{G}_{k,n})}^2 = 0$. This can be expressed as

$$\begin{aligned} 0 &= \lim_{j \rightarrow \infty} \int_{\eta \in G_{k,n}} \int_{\eta^\perp} |g(\eta, x'') - P\varphi_j(\eta, x'')|^2 dx'' d\eta \\ &= \lim_{j \rightarrow \infty} \int_{G_{k,n}} \|g(\eta, x'') - P\varphi_j(\eta, x'')\|_{L^2(\eta^\perp)}^2 d\eta. \end{aligned}$$

By the Plancherel theorem for the Fourier transform on $\eta^\perp$ the last integral is the same as $\int_{G_{k,n}} \|\widehat{g}(\eta, \xi'') - \widehat{P\varphi_j}(\eta, \xi'')\|_{L^2(\eta^\perp)}^2 d\eta$ which equals $\|\widehat{g} - \widehat{P\varphi_j}\|_{L^2(\mathcal{G}_{k,n})}^2$. Therefore, $\widehat{P\varphi_j} \rightarrow \widehat{g}$ in the L^2 norm on $\mathcal{G}_{k,n}$ and, hence, there is an appropriate subsequence such that

$$\widehat{P\varphi_j}(\eta, \xi'') \rightarrow \widehat{g}(\eta, \xi'') \quad (3.78)$$

almost everywhere on $\mathcal{G}_{k,n}$.

Since $\varphi_j \rightarrow f$ in the L^2 norm on $\mathbb{R}^n$, then by Plancherel we also have $\widehat{\varphi_j} \rightarrow \widehat{f}$ in L^2 norm. Since $|\widehat{\varphi_j}(\xi) - \widehat{f}(\xi)|^2$ is integrable, we can use the generalized polar

coordinate theorem to show that there is a constant c such that $0 = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^n} |\widehat{\varphi}_j(\xi) - \widehat{f}(\xi)|^2 d\xi = c \lim_{j \rightarrow \infty} \int_{\eta \in G_{k,n}} \int_{\eta^\perp} |\xi''|^k |\widehat{\varphi}_j(\xi'') - \widehat{f}(\xi'')|^2 d\xi''$. This means that $|\xi''|^k |\widehat{\varphi}_j(\xi'') - \widehat{f}(\xi'')|^2 \rightarrow 0$ in the L^2 norm on $\mathcal{G}_{k,n}$. Hence, there is an appropriate subsequence such that $|\xi''|^k |\widehat{\varphi}_j(\xi'') - \widehat{f}(\xi'')|^2 \rightarrow 0$ almost everywhere on $\mathcal{G}_{k,n}$. If we take a point $(\eta, \xi'') \in \mathcal{G}_{k,n}$, such that $\xi'' \neq 0$ and such that the previous limit exists, then because $|\xi''|^k$ is a nonzero constant with respect to the limit we will have $|\widehat{\varphi}_j(\xi'') - \widehat{f}(\xi'')|^2 \rightarrow 0$ and, hence, $\widehat{\varphi}_j(\xi'') \rightarrow \widehat{f}(\xi'')$ for almost all ξ'' in almost all $\eta^\perp$. Because φ_j is integrable, we can use the standard slice-projection theorem to get $\widehat{P\varphi_j}(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{\varphi}_j(\xi'')$ and hence we see that

$$\widehat{P\varphi_j}(\eta, \xi'') \rightarrow (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'') \quad (3.79)$$

for almost all ξ'' in almost all $\eta^\perp$.

If we now take any (η, ξ'') such that the limits in equations (3.78) and (3.79) both exist, then we see that the sequences involved are the same and hence the limits $\widehat{g}(\eta, \xi'')$ and $(2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')$ are the same. Since this is true for almost all $(\eta, \xi'') \in \mathcal{G}_{k,n}$, and since $g = \overline{P}f$, then the theorem is proved. ■

With the slice-projection theorem we can easily characterize the domain of $\overline{P}$.

Theorem 3.95. $D_{\overline{P}} = \mathcal{D}_{-\frac{k}{2}}$.

Proof. We start by showing that $D_{\overline{P}} \subset \mathcal{D}_{-\frac{k}{2}}$. By the generalized polar coordinate theorem we have

$$\int_{\mathbb{R}^n} \left(|\xi|^{-\frac{k}{2}} |\widehat{f}(\xi)| \right)^2 d\xi = \frac{1}{|G_{k,n-1}|} \int_{G_{k,n}} \int_{\eta^\perp} |\widehat{f}(\xi'')|^2 d\xi'' d\eta.$$

Now use theorem 3.94, the extended slice-projection theorem to get this result equal to

$$\frac{1}{(2\pi)^{\frac{k}{2}} |G_{k,n-1}|} \int_{G_{k,n}} \int_{\eta^\perp} |\overline{P}f(\eta, \xi'')|^2 d\xi'' d\eta < \infty.$$

The result is finite since $f \in D_{\overline{P}}$ implies $\overline{P}f \in L^2(\mathcal{G}_{k,n})$. Hence, $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$. Because we already know that $f \in L^2(\mathbb{R}^n)$ then it follows that $f \in \mathcal{D}_{-\frac{k}{2}}(\mathbb{R}^n)$.

Conversely, assume that $f \in \mathcal{D}_{-\frac{k}{2}}$. By lemma 3.91, there exists a sequence φ_j of rapidly decreasing functions such that $\varphi_j \rightarrow f$ and $|\xi|^{-\frac{k}{2}} \widehat{\varphi_j}(\xi) \rightarrow |\xi|^{-\frac{k}{2}} \widehat{f}(\xi)$ (in L^2 norm).

By Plancherel and slice-projection,

$$\|P\varphi_j - P\varphi_m\|_{L^2(G_{k,n})}^2 = (2\pi)^k \int_{G_{k,n}} \int_{\eta^\perp} |\widehat{\varphi_j}(\xi'') - \widehat{\varphi_m}(\xi'')|^2 d\xi'' d\eta.$$

Multiplying by $|\xi''|^k$ and its reciprocal in the integral and using the generalized polar coordinate theorem makes the previous expression equal to

$$\begin{aligned} & \int_{G_{k,n}} \int_{\eta^\perp} |\xi''|^k \left(|\xi''|^{-\frac{k}{2}} (\widehat{\varphi}_j(\xi'') - \widehat{\varphi}_m(\xi'')) \right)^2 d\xi'' d\eta \\ &= c \int_{\mathbb{R}^n} \left(|\xi|^{-\frac{k}{2}} |\widehat{\varphi}_j(\xi) - \widehat{\varphi}_m(\xi)| \right)^2 d\xi. \end{aligned}$$

This means that

$$\|P\varphi_j - P\varphi_m\|_{L^2(G_{k,n})}^2 = c \left\| |\xi|^{-\frac{k}{2}} \widehat{\varphi}_j(\xi) - |\xi|^{-\frac{k}{2}} \widehat{\varphi}_m(\xi) \right\|_{L^2(\mathbb{R}^n)}^2$$

But the convergence of $|\xi|^{-\frac{k}{2}} \widehat{\varphi}_j(\xi) \rightarrow |\xi|^{-\frac{k}{2}} \widehat{f}(\xi)$ in L^2 norm implies that $|\xi|^{-\frac{k}{2}} \widehat{\varphi}_j(\xi)$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. Hence, $P\varphi_j$ is a Cauchy sequence in $L^2(G_{k,n})$. This is enough to show that $f \in D_{\overline{P}}$. ■

Corollary 3.96. *The Schwartz class $\mathcal{S}$ of rapidly decreasing functions is contained in the domain of the extended k -plane transform for all k , $0 < k < n$. In general, functions in $L^1 \cap L^p$, $2 \leq p < \infty$ are contained in the domain of $\overline{P}$. For all these functions the extended k -plane transform agrees with the usual one and we can compute $\overline{P}f = Pf$ as an integral in the usual way.*

Proof. We first prove the result in the case that $f \in L^1 \cap L^p$, $2 \leq p < \infty$. The proof of corollary 3.76 shows that $f \in L^2$. When $0 < k < n$, $|\xi|^{-k}$ is integrable near the origin. Also $f \in L^1$ allows the use of the Riemann–Lebesgue lemma to conclude that $|\widehat{f}(\xi)|^2$ is a bounded function. Hence, $|\xi|^{-k} |\widehat{f}(\xi)|^2$ is integrable near the origin. Away from the origin, $\widehat{f}$ is L^2 and $|\xi|^{-\frac{k}{2}}$ is bounded. Therefore, $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi) \in L^2$ and hence $f \in \mathcal{D}_{-\frac{k}{2}} = D_{\overline{P}}$.

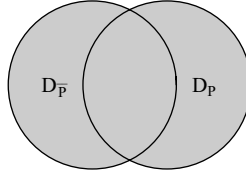
The proof of corollary 3.76 also shows that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ and, hence, the usual k -plane transform exists. By approximating f by rapidly decreasing functions it is easy to show that the extended k -plane transform equals the usual one.

The Schwartz class case follows as a special case of this result. ■

We defined the extended k -plane transform by extending the usual k -plane transform from rapidly decreasing functions to a larger subspace of $L^2(\mathbb{R}^n)$. The importance of this corollary is that it shows that this extension applies to a large class of functions that commonly arise in tomography. This includes all continuous functions of bounded support. It also includes the so-called “pixel” functions (see chapter 2, discussion preceding theorem 2.17).

However, there are functions for which the usual k -plane transform is defined and the extended k -plane transform is not. Also there are functions for which the extended k -plane transform is defined but the usual k -plane transform is not. If we let D_P denote the set of functions f such that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, then we know that D_P is essentially the largest space of functions defined on $\mathbb{R}^n$ such that we can define the

k -plane transform of f by convergent integrals. As above $D_{\overline{P}}$ is the domain of the extended k -plane transform. We can picture the relationship between these sets by the following diagram.



The following discussion explains these relationships.

There are functions which are in D_P but not in $D_{\overline{P}}$. Let k be an integer in the range $0 < k < \frac{n}{2}$. In this case it is very easy to find such a function; we define $f(x) = (1 + |x|)^{-\frac{1}{2}(k + \frac{n}{2})}$. It is easy to check that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. We can also check that f is not square integrable and hence is not in the domain of $\overline{P}$. More generally, if k is any integer in the range $0 < k < n$, then we can adapt Stein's construction (following corollary 3.86). The original construction yielded a function such that $(1 + |x|)^{k-n} f(x) \in L^1$ but such that $|\xi|^{-k} \widehat{f}(\xi)$ is not locally integrable. A close look at the construction shows that there is no loss in generality by assuming that $|\widehat{f}(\xi)| > 1$ on the unit ball at the origin. Therefore we have

$$\left\| |\xi|^{-\frac{k}{2}} \widehat{f}(\xi) \right\|_{L^2}^2 \geq \int_{B(1)} |\xi|^{-k} |\widehat{f}(\xi)|^2 d\xi \geq \int_{B(1)} |\xi|^{-k} |\widehat{f}(\xi)| d\xi = \infty$$

Therefore $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi) \notin L^2$ and thus we have constructed a function f which is not in $D_{\overline{P}}$ but which is in D_P .

In the other direction, Smith and Solmon constructed an example of a function which is in $D_{\overline{P}}$ but not in D_P . This may be found on page 544 of [565].

We have already shown that the intersection of D_P and $D_{\overline{P}}$ contains the most common functions used in tomography, such as Schwartz class functions and functions in $L^1 \cap L^p$.

Theorem 3.97 (Isometry theorem). *If $f \in D_{\overline{P}}$, then there is a constant c such that*

$$c \left\| \Lambda^{\frac{k}{2}+s} \overline{P} f \right\|_{L^2(\mathcal{G}_{k,n})} = \left\| \Lambda^s f \right\|_{L^2(\mathbb{R}^n)}. \quad (3.80)$$

The constant is

$$c = \frac{1}{\sqrt{(2\pi)^k |G_{k,n-1}|}}$$

Furthermore, if $f, g \in D_{\overline{P}}$, then

$$c^2 \left\langle \Lambda^{\frac{k}{2}+s} \overline{P} f, \Lambda^{\frac{k}{2}+s} \overline{P} g \right\rangle_{L^2(\mathcal{G}_{k,n})} = \langle \Lambda^s f, \Lambda^s g \rangle_{L^2(\mathbb{R}^n)}.$$

Proof. From the definition of the lambda operator and from the extended slice-projection theorem we get

$$\left| \left(\Lambda^{\frac{k}{2}+s} \overline{P} f \right)^\wedge (\eta, \xi'') \right|^2 = (2\pi)^k |\xi''|^{k+2s} |\widehat{f}(\xi'')|^2.$$

Integrating this over $\mathcal{G}_{k,n}$ and using Plancherel's theorem makes the left-hand side equal to $\|\Lambda^{\frac{k}{2}+s} \overline{P} f\|_{L^2(\mathcal{G}_{k,n})}^2$. On the right-hand side we write the integrand as $|\xi''|^k [(2\pi)^k |\xi''|^{2s} |\widehat{f}(\xi'')|^2]$ and we use the generalized polar coordinate theorem to get

$$\begin{aligned} \left\| \Lambda^{\frac{k}{2}+s} \overline{P} f \right\|_{L^2(\mathcal{G}_{k,n})}^2 &= (2\pi)^k |G_{k,n-1}| \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \\ &= (2\pi)^k |G_{k,n-1}| \left\| \Lambda^s f \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

This proves equation (3.80). The relation for the inner product is now a direct consequence of the polar identity on inner product spaces. ■

Remark 3.98. The isometry theorem is a generalization of theorem 1.4 of Ludwig [405], which has a statement and proof of this result in the case $k = n - 1, s = 0$. Also see Smith, Solmon and Wagner [566].

Corollary 3.99. *If $f \in D_{\overline{P}} \cap \mathcal{D}_t$, then $\overline{P} f \in \mathcal{D}_{t+\frac{k}{2}}$.*

Proof. Since $D_{\overline{P}} = \mathcal{D}_{-\frac{k}{2}}$, we can use the isometry theorem with $s = -\frac{k}{2}$ and this shows that $\overline{P} f \in L^2(\mathcal{G}_{k,n})$. We can also apply the isometry theorem with $s = t$, which shows that $\Lambda^{\frac{k}{2}+s} \overline{P} f \in L^2(\mathcal{G}_{k,n})$ and this proves it. ■

Remark 3.100. Intuitively speaking, this corollary states that the Radon transform or k -plane transform of a function is smoother than the function itself. For example, if t is positive and $f \in D_{\overline{P}} \cap \mathcal{D}_t$, then f has derivatives up to order m , where m is the greatest integer less than or equal to t . This is because $\mathcal{D}_t$ is a Sobolev space if $t \geq 0$. The corollary then shows that $\overline{P} f(\eta, x'')$ has about $\frac{k}{2}$ more derivatives in the x'' variable. Compare Strichartz [584].

If V is an isometry of Hilbert spaces, then $V^* V = V V^* = I$, so V is invertible and V^* is its inverse. Taking $s = 0$ in the isometry theorem shows that $V = c \Lambda^{\frac{k}{2}} \overline{P}$ is an isometry, provided that we restrict to the subspace $D_{\overline{P}}$. It is easily shown that $\Lambda^{\frac{k}{2}}$ is self-adjoint so we would expect that

$$\begin{aligned} I &= \left(c \Lambda^{\frac{k}{2}} \overline{P} \right)^* \left(c \Lambda^{\frac{k}{2}} \overline{P} \right) = c^2 \overline{P}^* \left(\Lambda^{\frac{k}{2}} \right)^* \Lambda^{\frac{k}{2}} \overline{P} \\ &= c^2 \overline{P}^* \left(\Lambda^{\frac{k}{2}} \right)^2 \overline{P} = \frac{1}{(2\pi)^k |G_{k,n-1}|} \overline{P}^* \Lambda^k \overline{P} \end{aligned}$$

and hence that we would have the inversion formula $\frac{1}{(2\pi)^k |G_{k,n-1}|} \overline{P}^* \Lambda^k \overline{P}(f) = f$ for $f \in D_{\overline{P}}$. Note the resemblance between this inversion formula and the previous inversion formula $\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f = f$ that we have established in this chapter for various spaces of functions.

Unfortunately, in general, $D_{\bar{P}}$ is not a Hilbert space so this argument is not valid. Nonetheless, the result turns out to be correct. Therefore, we next investigate the adjoint to the unbounded operator $\bar{P}$ and then prove the correctness of this inversion formula.

Let $A : H \rightarrow G$ be a densely defined linear operator from a Hilbert space H to another Hilbert space G . Let D_A be the domain of this operator. If D_A is equal to H and if A is bounded, then it is well known that there is a unique adjoint operator $A^* : G \rightarrow H$ such that $\langle Ax, y \rangle_G = \langle x, A^*y \rangle_H$ for all $x \in H$ and $y \in G$.

To generalize the idea of the adjoint to operators which are unbounded we provide the following definition.

Definition 3.101. If $A : H \rightarrow G$ is a densely defined, but possibly unbounded, linear operator, then the definition of the adjoint is generalized to take this form: the domain of the adjoint operator D_{A^*} is defined to be the set of elements $y \in G$, such that the functional $x \mapsto \langle Ax, y \rangle_G, x \in D_A$ is bounded. Since D_A is dense in H and since H is a Hilbert space, then this functional extends to a continuous functional on H and a unique point $x^* \in H$ exists such that the functional just described is equal to the functional $x \mapsto \langle x, x^* \rangle_H$. Then we define $A^*y = x^*$ and the equation

$$\langle Ax, y \rangle_G = \langle x, A^*y \rangle_H \quad (3.81)$$

is therefore valid for all $x \in D_A$ and $y \in D_{A^*}$.

Remark 3.102. In checking whether y is in the domain of the adjoint, we can check that the functional $x \mapsto \langle Ax, y \rangle_G$ is continuous on the complete domain D_A or on a dense subspace. For example, it is most convenient to use the Schwartz class $\mathcal{S}$ in checking whether a function g is in the domain of $\bar{P}^*$. However, the relation (3.81) is valid on the complete domain of the extended operator.

Remark 3.103. The formal adjoint theorem (theorem 3.29) states that if f is a non-negative measurable function on $\mathbb{R}^n$ and g is a nonnegative measurable function on $\mathcal{G}_{k,n}$, then

$$\left\langle f, P^\# g \right\rangle_{\mathbb{R}^n} = \langle Pf, g \rangle_{G_{k,n}}$$

whereas the relation (3.81) in the case $A = \bar{P}$ states that

$$\left\langle f, \bar{P}^* g \right\rangle_{L^2(\mathbb{R}^n)} = \langle \bar{P}f, g \rangle_{L^2(\mathcal{G}_{k,n})}$$

where $f \in D_{\bar{P}}$ and $g \in D_{\bar{P}^*}$. The next result examines this relationship more closely by giving the exact conditions under which $P^\# g = \bar{P}^* g$.

Theorem 3.104. If $g \in L^2(\mathbb{R}^n)$, then $P^\# g$ is locally square integrable.

Furthermore, the domain of the adjoint $\bar{P}^*$ of the extended k -plane transform $\bar{P}$ consists of all square integrable functions g defined on $\mathcal{G}_{k,n}$, such that $P^\# g \in L^2(\mathbb{R}^n)$.

Finally, if g is in the domain of $\bar{P}^*$, then $\bar{P}^* g = P^\# g$.

Proof. For the first statement we merely remark that the proof is similar to the demonstration that $P^\# g$ is locally integrable if $g \in L^1(\mathcal{G}_{k,n})$ (see corollary 3.30).

We next show that $D_{\overline{P}^*}$ is a subset of the set of all square integrable functions g defined on $\mathcal{G}_{k,n}$, such that $P^\# g \in L^2(\mathbb{R}^n)$. If g is in the domain of $\overline{P}^*$, then $g \in L^2$ and there is a constant C , such that

$$|\langle P\varphi, g \rangle| \leq C \|\varphi\|_{L^2(\mathbb{R}^n)} \quad (3.82)$$

for all rapidly decreasing functions φ . Because $P^\# g$ is locally square integrable, then it is square integrable on the ball $B(r)$ of radius r , where r is any positive real number. We now proceed to show that $\|P^\# g(x)\|_{L^2(B(r))} \leq C$ for any $r > 0$.

For now fix r . The desired inequality is obvious if $\|P^\# g(x)\|_{L^2(B(r))} = 0$, so we can assume that $\|P^\# g(x)\|_{L^2(B(r))} \neq 0$.

For convenience define $N_r = \|P^\# g\|_{L^2(B(r))}$. Let ε be an arbitrary positive number. Because $P^\# g(x)$ is square integrable on $B(r)$, then there exists a function $\varphi_\varepsilon \in C_0^\infty(B(r))$ such that

$$\|P^\# g(x) - \varphi_\varepsilon\|_{L^2(B(r))} < \min \left\{ \frac{\varepsilon}{2C}, \frac{\varepsilon}{2N_r} \right\}. \quad (3.83)$$

We then have

$$\begin{aligned} N_r^2 &= \|P^\# g\|_{L^2(B(r))}^2 = \left| \langle P^\# g, P^\# g \rangle_{L^2(B(r))} \right| \\ &\leq \left| \langle P^\# g - \varphi_\varepsilon, P^\# g \rangle_{L^2(B(r))} \right| + \left| \langle \varphi_\varepsilon, P^\# g \rangle_{L^2(B(r))} \right| \end{aligned} \quad (3.84)$$

But by the Cauchy–Schwarz inequality and (3.83) we have

$$\begin{aligned} \left| \langle P^\# g - \varphi_\varepsilon, P^\# g \rangle_{L^2(B(r))} \right| &\leq \|P^\# g - \varphi_\varepsilon\|_{L^2(B(r))} \|P^\# g\|_{L^2(B(r))} \\ &< \frac{\varepsilon}{2N_r} N_r = \frac{\varepsilon}{2} \end{aligned} \quad (3.85)$$

Also, by the formal adjoint theorem

$$\left| \langle \varphi_\varepsilon, P^\# g \rangle_{L^2(B(r))} \right| = \left| \langle P\varphi_\varepsilon, g \rangle_{L^2(\mathcal{G}_{k,n})} \right| \leq C \|\varphi_\varepsilon\|_{L^2(\mathbb{R}^n)}. \quad (3.86)$$

This is by inequality (3.82) and the fact that C^∞ functions with support in $B(r)$ are rapidly decreasing and satisfy $\|\varphi_\varepsilon\|_{L^2(\mathbb{R}^n)} = \|\varphi_\varepsilon\|_{L^2(B(r))}$. But inequality (3.83) implies

$$\begin{aligned} \|\varphi_\varepsilon\|_{L^2(B(r))} &< \|P^\# g(x)\|_{L^2(B(r))} + \frac{\varepsilon}{2C} \\ &= N_r + \frac{\varepsilon}{2C} \end{aligned} \quad (3.87)$$

Putting together inequalities (3.84)–(3.87) we get

$$\begin{aligned} N_r^2 &\leq \frac{\varepsilon}{2} + C \left(N_r + \frac{\varepsilon}{2C} \right) \\ &= C N_r + \varepsilon. \end{aligned}$$

Because this is true for all $\varepsilon > 0$ and since $N_r \neq 0$, then $N_r \leq C$. Since $N_r = \|P^\# g(x)\|_{L^2(B(r))}$ this implies that $\|P^\# g(x)\|_{L^2(B(r))} \leq C$ for any $r > 0$. In particular, this implies that $\int_{|x| < r} |P^\# g(x)|^2 dx \leq C^2$ for all $r > 0$. The functions $|P^\# g(x)|^2$ restricted to $|x| < r$ are an increasing sequence of positive integrable functions with integrals bounded by C^2 . By the monotone convergence theorem we then have

$$\begin{aligned} \|P^\# g(x)\|_{L^2(\mathbb{R}^n)}^2 &= \lim_{r \rightarrow \infty} \int_{|x| < r} |P^\# g(x)|^2 dx \\ &\leq C^2 < \infty \end{aligned}$$

This implies that $P^\# g$ is square integrable.

Conversely, assume that $g \in L^2$ and $P^\# g \in L^2(\mathbb{R}^n)$. Then the Cauchy–Schwarz inequality implies that $f \mapsto \langle f, P^\# g \rangle$, $f \in D_{\overline{P}}$ is a bounded linear functional. As a consequence g is in the domain of $\overline{P}^*$.

For the last statement assume that g is in the domain of $\overline{P}^*$. As we now know, this implies $P^\# g \in L^2$, so, as we just showed, the functional $f \mapsto \langle f, P^\# g \rangle$ is bounded. The definition of the adjoint shows that $\overline{P}^* g = P^\# g$. ■

Remark 3.105. We will prove in theorem 3.114, that if $k \geq \frac{n}{2}$, then the domain of $\overline{P}^*$ does not contain a single nontrivial nonnegative function. Then the only way a function g can be in the domain of P^* is for there to be enough cancellations in the integral defining its backprojection so that $P^\# g$ decays fast enough at ∞ to be square integrable. In the case of the Radon transform, any odd function g has this property.

Theorem 3.106 (Inversion of the extended k -plane transform; Solmon [569]). *Let P be the k -plane transform. For any real number s , if $f \in \mathcal{D}_{s-k} \cap \mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$, then $\Lambda^s \overline{P} f$ is in the domain of $\overline{P}^*$ and we have the inversion formula*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^{k-s} P^\# \Lambda^s \overline{P} f = f. \quad (3.88)$$

In particular:

1. If f is in the domain of $\overline{P}$, then we have the inversion formula

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^{\frac{k}{2}} P^\# \Lambda^{\frac{k}{2}} \overline{P} f = f.$$

2. If $s \geq \frac{k}{2}$ and $f \in \mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$, then $\Lambda^s \overline{P} f$ is in the domain of $\overline{P}^*$ and the inversion formula (3.88) holds.
3. If $s < \frac{k}{2}$ and $f \in \mathcal{D}_{s-k}$, then $\Lambda^s \overline{P} f$ is in the domain of $\overline{P}^*$ and the inversion formula (3.88) holds.
4. If $\varphi \in \mathcal{S}$, then $\Lambda^k P \varphi$ is in the domain of $\overline{P}^*$ and $\frac{1}{(2\pi)^k |G_{k,n-1}|} P^\# \Lambda^k P \varphi = \varphi$.

In general, the inversion formula (3.88) is valid for rapidly decreasing functions provided that $s > k - \frac{n}{2}$.

5. If $f \in L^2(\mathbb{R}^n)$ and $|\xi|^{-k} \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, then $\overline{P}f$ is in the domain of $\overline{P}^*$ and $\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# \overline{P}f = f$.

Proof. For the main result let $g = \overline{P}f$. Then g is well defined since $f \in \mathcal{D}_{-\frac{k}{2}} = \text{domain}(\overline{P})$.

We remind the reader that as a consequence of the definition of the extended k -plane transform any rapidly decreasing function φ is in the domain of $\overline{P}$ and by corollary 3.99, $P\varphi = \overline{P}\varphi \in L^2(\mathcal{G}_{k,n})$. By the hypothesis that $f \in \mathcal{D}_{-\frac{k}{2}}$ and the isometry theorem, we have $\Lambda^s g \in L^2(\mathcal{G}_{k,n})$. By the hypothesis that $f \in \mathcal{D}_{s-k}$ we have $\Lambda^{s-k} f \in L^2(\mathbb{R}^n)$. Hence, the inner products in the following equation are well defined and finite and we will prove momentarily that for any rapidly decreasing function φ

$$\langle P\varphi, \Lambda^s g \rangle_{L^2(\mathcal{G}_{k,n})} = \left\langle \varphi, (2\pi)^k |G_{k,n-1}| \Lambda^{s-k} f \right\rangle_{L^2(\mathbb{R}^n)}. \quad (3.89)$$

We now proceed to prove this equation: since $\Lambda^s g \in L^2(\mathcal{G}_{k,n})$, then $\Lambda^s g(\eta, x'')$ is in $L^2(\eta^\perp)$ for almost every $\eta \in G_{k,n}$ and this allows us to use Parseval's theorem applied to the $(n-k)$ -planes $\eta^\perp$ in the next calculation:

$$\begin{aligned} \langle P\varphi, \Lambda^s g \rangle_{L^2(\mathcal{G}_{k,n})} &= \int_{G_{k,n}} \int_{\eta^\perp} P\varphi(\eta, x'') \overline{\Lambda^s g(\eta, x'')} dx'' d\eta \\ &= \int_{G_{k,n}} \langle P\varphi(\eta, x''), \Lambda^s g(\eta, x'') \rangle_{L^2(\eta^\perp)} d\eta \\ &= \int_{G_{k,n}} \langle \widehat{P\varphi}(\eta, \xi''), \widehat{\Lambda^s g}(\eta, \xi'') \rangle_{L^2(\eta^\perp)} d\eta \\ &= (2\pi)^{\frac{k}{2}} \int_{G_{k,n}} \int_{\eta^\perp} \widehat{\varphi}(\xi'') \overline{|\xi''|^{s-k} \widehat{f}(\xi'')} d\xi''. \end{aligned}$$

Because $\varphi \in \mathcal{S}$ it was valid to use the slice-projection theorem on $\widehat{P\varphi}(\eta, \xi'')$ in the last step. The fact that $f \in \mathcal{D}_{-\frac{k}{2}}$ allows the use of the extended slice-projection theorem to get $\widehat{g}(\eta, \xi'') = (\overline{P}f)^\wedge(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')$ for almost every ξ'' in almost every $(n-k)$ -plane $\eta^\perp$. Using this in the previous equation along with the generalized polar coordinate theorem gives

$$\begin{aligned} \langle P\varphi, \Lambda^s g \rangle_{L^2(\mathcal{G}_{k,n})} &= (2\pi)^k \int_{G_{k,n}} \int_{\eta^\perp} |\xi''|^k \widehat{\varphi}(\xi'') \overline{|\xi''|^{s-k} \widehat{f}(\xi'')} d\xi'' \quad (3.90) \\ &= (2\pi)^k |G_{k,n-1}| \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) \overline{|\xi|^{s-k} \widehat{f}(\xi)} d\xi \\ &= (2\pi)^k |G_{k,n-1}| \left\langle \widehat{\varphi}(\xi), |\xi|^{s-k} \widehat{f}(\xi) \right\rangle_{L^2(\mathbb{R}^n)} \end{aligned}$$

Note that $f \in \mathcal{D}_{s-k}$ so $\Lambda^{s-k} f \in L^2$ and taking the Fourier transform we get $|\xi|^{s-k} \widehat{f}(\xi) = \widehat{\Lambda^{s-k} f}$. We now use the Parseval theorem on $\mathbb{R}^n$ to show that

$$\left\langle \widehat{\varphi}(\xi), |\xi|^{s-k} \widehat{f}(\xi) \right\rangle_{L^2(\mathbb{R}^n)} = \left\langle \varphi, \Lambda^{s-k} f \right\rangle_{L^2(\mathbb{R}^n)},$$

we substitute this result in equation (3.90) and we see that the proof of equation (3.89) is complete.

Since $\Lambda^{s-k} f \in L^2$, the functional $\varphi \mapsto \langle \varphi, \Lambda^{s-k} f \rangle_{L^2(\mathbb{R}^n)}$ is bounded. By (3.89) the functional $\varphi \mapsto \langle P\varphi, \Lambda^s g \rangle_{L^2(\mathcal{G}_{k,n})}$ is also bounded, and hence $\Lambda^s g$ is in the domain of the adjoint of $\overline{P}$. Furthermore, the definition of the adjoint and equation (3.89) show that

$$\overline{P}^* (\Lambda^s g) = (2\pi)^k |G_{k,n-1}| \Lambda^{s-k} f. \quad (3.91)$$

A consequence of this equation is that

$$|\xi|^{k-s} \left(\overline{P}^* (\Lambda^s g) \right)^\wedge (\xi) = c |\xi|^{k-s} \left(\Lambda^{s-k} f \right)^\wedge (\xi) = c \widehat{f}(\xi).$$

But $f \in L^2$ so $\widehat{f} \in L^2$. Hence, $|\xi|^{k-s} \left(\overline{P}^* (\Lambda^s g) \right)^\wedge (\xi) \in L^2$ and we can therefore apply Λ^{k-s} , which is the inverse of Λ^{s-k} , to equation (3.91). This gives the desired inversion formula (3.88) once we take into account the fact that $\overline{P}^* = P^\#$ on the domain of the adjoint.

If the hypothesis of conclusion 1 holds, and if we take $s = \frac{k}{2}$, then the hypothesis $f \in \mathcal{D}_{s-k} \cap \mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$ of the main result becomes $f \in \mathcal{D}_{-\frac{k}{2}} \cap \mathcal{D}_0 \cap \mathcal{D}_{-\frac{k}{2}}$, which of course simplifies to $\mathcal{D}_{-\frac{k}{2}}$, which is the domain of $\overline{P}$. Thus, the main result applies and gives conclusion 1.

In both conclusions 2 and 3 we have $f \in \mathcal{D}_{\overline{P}} = \mathcal{D}_{-\frac{k}{2}}$. In the first case, this is obvious. In the case that $s < \frac{k}{2}$, then $s - k < -\frac{k}{2}$. On the set where $|\xi| < 1$ this implies that $|\xi|^{-\frac{k}{2}} |\widehat{f}(\xi)| \leq |\xi|^{s-k} |\widehat{f}(\xi)|$, this together with the fact that $f \in \mathcal{D}_{s-k}$ shows that $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi)$ is L^2 on the set $\{|\xi| < 1\}$. On the set where $|\xi| \geq 1$, we have $|\xi|^{-\frac{k}{2}} \leq 1$ and a similar calculation shows that $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi)$ is L^2 on this set because $f \in \mathcal{D}_{s-k}$ also implies $f \in L^2$. Hence, $f \in L^2(\mathbb{R}^n)$ and $|\xi|^{-\frac{k}{2}} \widehat{f}(\xi) \in L^2(\mathbb{R}^n)$, so $f \in \mathcal{D}_{-\frac{k}{2}}$.

For conclusion 2, we have the hypothesis $s \geq \frac{k}{2}$ and $f \in \mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$. In this case note that $\mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}} \subset \mathcal{D}_{s-k}$. To see this decompose $f = f_1 + f_2$, where f_1 is the restriction of f to the unit ball and f_2 is the restriction to the complement. Since $s \geq \frac{k}{2}$, then $s - k \geq -\frac{k}{2}$. A calculation similar to the one in the preceding paragraph shows that $|\xi|^{s-k} |\widehat{f}_1(\xi)|$ and $|\xi|^{s-k} |\widehat{f}_2(\xi)|$ are in L^2 and, hence, that $f \in \mathcal{D}_{s-k}$. The hypotheses of the main result now apply and this case is finished.

For conclusion 3 we have the hypothesis $s < \frac{k}{2}$ and $f \in \mathcal{D}_{s-k}$. The same sort of decomposition arguments used above proves that f is also in $\mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$. Hence, we may apply the main result to finish this case.

By theorem 3.89 it is easy to show that if $\varphi \in \mathcal{S}$, then $\varphi \in \mathcal{D}_{s-k} \cap \mathcal{D}_{s-\frac{k}{2}} \cap \mathcal{D}_{-\frac{k}{2}}$ for $s > k - \frac{n}{2}$. The second part of conclusion 4 follows from the main part of the theorem. The first part of conclusion 4 now follows by setting $s = k$.

Conclusion 5 is exactly conclusion 3 with $s = 0$. ■

Corollary 3.107. $\overline{P}(\mathcal{D}_{-k}) \subset \text{domain}(\overline{P}^*)$ and $\overline{P}^*$ is injective on $\overline{P}(\mathcal{D}_{-k})$.

Proof. The inclusion follows directly from conclusion 3 of the theorem with $s = 0$.

To prove the injectivity, suppose that g is in $\overline{P}(\mathcal{D}_{-k})$ so that there exists a function $f \in \mathcal{D}_{-k}$ with $g = \overline{P}f$. Note that by the first part of the corollary it is valid to apply $\overline{P}^*$ to g . If $\overline{P}^*g = 0$, then $\|g\|_{L^2(\mathcal{G}_{k,n})}^2 = \langle g, g \rangle = \langle \overline{P}f, g \rangle = \langle f, \overline{P}^*g \rangle = 0$. Hence, $\overline{P}^*g = 0$ implies $g = 0$. ■

No two distinct functions in the domain of the extended k -plane transform can have the same extended k -plane transform as the next result shows.

Theorem 3.108. $\overline{P}$ is injective on $\mathcal{D}_{-\frac{k}{2}}$.

Proof. Let $f \in \mathcal{D}_{-\frac{k}{2}}$ with $\overline{P}f = 0$. Because f is in the domain of $\overline{P}$, a sequence of rapidly decreasing functions (φ_j) exists with $f = \lim \varphi_j$ and $\overline{P}f = \lim \overline{P}\varphi_j$. Then $\|f\|^2 = \langle f, f \rangle = \lim \langle f, \varphi_j \rangle$. We can apply the inversion theorem to write $\varphi_j = c\overline{P}^* \Lambda^k P \varphi_j$ and this and the previous equation give

$$\|f\|^2 = \lim \langle f, c\overline{P}^* \Lambda^k P \varphi_j \rangle = c \lim \langle \overline{P}f, \Lambda^k P \varphi_j \rangle = 0$$

Therefore $f = 0$ and $\overline{P}$ is injective on $\mathcal{D}_{-\frac{k}{2}}$. ■

3.10 The Action of the k -Plane Transform on L^p Functions

All functions arising in the applications of tomography are compactly supported measurable functions.⁵ Hence, these functions lie in any L^p space. We already have several results concerning these types of functions in the L^2 and L^1 cases.

However, there are good reasons to study the inversion and mapping properties of general L^p functions. For example, Strichartz [584] page 699 states: “There are two reasons why we believe this study is important. The first is that we would like to know how the size of a function influences the size and smoothness of its Radon transform, and the L^p spaces furnish a fairly refined measurement of size. The second is that the Radon transform is a natural test case to explore the powers and limitations of our technical understanding of various aspects of harmonic analysis on Euclidean and non-Euclidean spaces.”

Therefore, in this section we will be interested in inversion formulas for the k -plane transform of L^p functions and we will be interested in L^p estimates for the k -plane transform (we use the term “ L^p estimate” as an abbreviation for an inequality showing that the k -plane transform is a bounded linear operator between certain L^p -spaces).

We already have some results in this direction. We established in corollary 3.25 that the k -plane transform is a bounded L^1 operator. We also established some inversion theorems for certain L^1 and L^2 functions.

We use the notation p' for the *conjugate index* of an integer p . The conjugate index is defined via the equation $\frac{1}{p} + \frac{1}{p'} = 1$. It is known that for any measure space X ,

⁵ Examples would be digital pictures or functions representing anatomical structures.

$L^{p'}(X)$ is the conjugate space of $L^p(X)$ for $1 \leq p \leq \infty$ (compare, section 3.2). In this section we always assume that p is an extended real number with $1 \leq p \leq \infty$.

One question that arises is whether the k -plane transform is defined for L^p functions. It is sufficient for f to be integrable. In general, we have shown that the condition $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ is sufficient for the k -plane transform to be well defined. This condition is also necessary for nonnegative functions.

If f is a compactly supported L^p function, then it is integrable and hence the k -plane transform exists on $L_0^p(\mathbb{R}^n)$. This can be seen by using Hölder's inequality with $f \in L_0^p(\mathbb{R}^n)$ and with g being the characteristic function of the support of f : it follows easily that $L_0^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.

If f is a general L^p function, possibly not compactly supported, then we will need the condition $(1 + |x|)^{k-n} f(x) \in L^1$.

In this section P will always denote the k -plane transform acting on $\mathbb{R}^n$.

We first deal with compactly supported L^p functions. After that we consider the more general case.

3.10.1 The Action of the k -Plane Transform on Compactly Supported L^p Functions

First, we have the following inversion theorem for the k -plane transform of compactly supported L^p functions.

Theorem 3.109 (Inversion theorem for L_0^p). *Let $f \in L_0^p(\mathbb{R}^n)$. Then the inversion formula $f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^k P^\# P f$ is valid under either of the following two conditions:*

- (1) $p \geq 2$, or
- (2) $1 \leq p < 2$, and $k > \frac{n}{p}$.

Proof. Consider the first case where $p \geq 2$. We showed in the introduction to this section that Hölder's inequality implies $f \in L^1(\mathbb{R}^n)$. But we also have $f \in L^2(\mathbb{R}^n)$. To show this, let $A = \{x \in \mathbb{R}^n : |f(x)| > 1\} \cap \text{supp}(f)$ and let $B = \{x \in \mathbb{R}^n : |f(x)| \leq 1\} \cap \text{supp}(f)$. Since $p \geq 2$, then $|f(x)|^2 \leq |f(x)|^p$ on A , and since $f \in L^p$, this implies that $f \in L^2(A)$. On the set B we have $|f(x)|^2 \leq 1$ and since B has a compact closure, then $\|f\|_{L^2(B)} < \infty$. Since A and B are disjoint and their union is the support of f , then we see that $f \in L^2$.

Hence, $f \in L^1 \cap L^2$ and corollary 3.81 applies to give the inversion formula.

Now consider the second case where $1 \leq p < 2$, and $k > \frac{n}{p}$. As in the first part of the proof, $f \in L^1(\mathbb{R}^n)$. We now show that $|\xi|^{-k} \widehat{f}(\xi)$ is in $L^1(\mathbb{R}^n)$. Let $U = \{\xi : |\xi| \leq 1\}$ and let $V = \{\xi : |\xi| > 1\}$. It is enough to show that $|\xi|^{-k} \widehat{f}(\xi)$ is integrable on each set. Since $f \in L^1$, then $\widehat{f}$ is a bounded function. Also $|\xi|^{-k}$ is locally integrable since $k < n$. Therefore, the product $|\xi|^{-k} \widehat{f}(\xi)$ is in $L^1(U)$. On the set V , where $|\xi| > 1$, it is an easy calculation to check that the condition $k > \frac{n}{p}$ implies $|\xi|^{-k} \in L^p(V)$. By the Hausdorff–Young inequality, $\widehat{f} \in L^{p'}$, so by Hölder's inequality the product $|\xi|^{-k} \widehat{f}(\xi)$ is in $L^1(V)$.

Having verified the hypotheses of theorem 3.44, we can conclude that the inversion formula is valid in this case. ■

Theorem 3.110. *If f has compact support and is in $L^p(\mathbb{R}^n)$, and if $\eta \in G_{k,n}$, then we have $P_\eta f \in L^p(\eta^\perp)$ and*

$$\|P_\eta f\|_{L^p(\eta^\perp)} \leq c (\delta(\text{supp}(f)))^{\frac{k}{p-1}} \|f\|_{L^p(\mathbb{R}^n)}$$

where c is a constant depending only on k and p .

Proof. Let $\eta \in G_{k,n}$. We can express any $x \in \mathbb{R}^n$ as $x = x' + x''$, where $x' \in \eta$ and $x'' \in \eta^\perp$.

Let g be the characteristic function of $\text{supp}(f)$. Then $|f(x' + x'')| = |f(x' + x'')|g(x' + x'')$ and hence

$$\begin{aligned} |Pf(\eta, x'')| &\leq \int_{\eta} |f(x' + x'')| g(x' + x'') dx' \\ &= \langle |f|, g \rangle_{\eta+x''} \end{aligned} \quad (3.92)$$

where we use the notation $\langle \cdot, \cdot \rangle_{\eta+x''}$ to denote the inner product on the measure space $\eta + x''$.

Since $f \in L^p(\mathbb{R}^n)$, then f restricted to $\eta + x''$ is in $L^p(\eta + x'')$ for almost all x'' , whereas g being the characteristic function of a compact set is in $L^{p'}(\eta + x'')$ for all x'' . Hence, we can apply Hölder's inequality on the measure space $\eta + x''$ to get

$$\langle |f|, g \rangle_{\eta+x''} \leq \|f\|_{L^p(\eta+x'')} \|g\|_{L^{p'}(\eta+x'')} \quad (3.93)$$

for all η and almost all x'' .

Now assume that $(\eta + x'') \cap \text{supp } f$ is nonempty. If we choose a k -dimensional ball B' in $\eta + x''$ centered at any point of the intersection $(\eta + x'') \cap \text{supp } f$ and of radius $\delta(\text{supp } f)$, then it is easy to check that $(\eta + x'') \cap \text{supp } f \subset B'$. Let v_1 denote the volume of a unit k -dimensional ball. Then v_1 is a constant depending only on k and $|B'| = v_1(\delta(\text{supp } f))^k$. Now let $c = v_1^{\frac{1}{p'}}$, and let $r = \delta(\text{supp } f)$. Then c is a constant depending only on k and p . Using this and the definition of g we get

$$\begin{aligned} \|g\|_{L^{p'}(\eta+x'')} &= \left[\int_{(\eta+x'') \cap \text{supp } f} dx' \right]^{\frac{1}{p'}} \\ &= |(\eta + x'') \cap \text{supp } f|^{\frac{1}{p'}} \leq |B'|^{\frac{1}{p'}} = cr^{\frac{k}{p'}} \end{aligned} \quad (3.94)$$

Using the results of (3.93) and (3.94) in (3.92) yields $|Pf(\eta, x'')| \leq cr^{\frac{k}{p'}} \|f\|_{L^p(\eta+x'')}$. This inequality assumed that $(\eta + x'') \cap \text{supp } f \neq \emptyset$. If the intersection is empty, then $Pf(\eta, x'') = 0$ and the inequality is still true. Hence we can integrate

the p th power of this inequality with respect to the measure dx'' of $\eta^\perp$ to obtain

$$\begin{aligned} \|P_\eta f\|_{L^p(\eta^\perp)}^p &= \int_{\eta^\perp} |Pf(\eta, x'')|^p dx'' \\ &\leq cr^{\frac{kp}{p'}} \int_{\eta^\perp} \|f\|_{L^p(\eta+x'')}^p dx'' \\ &= cr^{\frac{kp}{p'}} \int_{\eta^\perp} \int_{\eta} |f(x' + x'')|^p dx' dx''. \end{aligned}$$

Apply equation (3.34) of corollary 3.25 to the last integral to obtain

$$\|P_\eta f\|_{L^p(\eta^\perp)}^p \leq cr^{\frac{kp}{p'}} \int_{\mathbb{R}^n} |f(x)|^p dx = cr^{\frac{kp}{p'}} \|f\|_{L^p(\mathbb{R}^n)}^p$$

Taking the p th root and using the definition of r and p' yields

$$\|P_\eta f\|_{L^p(\eta^\perp)} \leq c (\delta(\text{supp}(f)))^{\frac{k}{p-1}} \|f\|_{L^p(\mathbb{R}^n)}$$

where c is a constant depending only on k and p . ■

Corollary 3.111. *Let K be a compact subset of $\mathbb{R}^n$. If $f \in L^p(K)$, then $Pf \in L^p(\mathcal{G}_{k,n})$. We have the “mixed norm” estimate:*

$$\begin{aligned} \|Pf\|_{L^p(\mathcal{G}_{k,n})}^p &= \left\| \|P_\eta f\|_{L^p(\eta^\perp)} \right\|_{L^p(\mathcal{G}_{k,n})}^p \\ &\leq C \|f\|_{L^p(K)}^p. \end{aligned}$$

As a consequence, the k -plane transform restricted to $L^p(K)$ is a bounded linear operator from $L^p(K)$ to $L^p(\mathcal{G}_{k,n})$. In general, the k -plane transform is an unbounded linear operator from $L_0^p(\mathbb{R}^n)$ to $L^p(\mathcal{G}_{k,n})$.

Proof. We use the previous theorem applied to functions $f \in L^p(K)$. Because all these functions have a common support, we see that there is a constant C such that

$$\int_{\eta^\perp} |Pf(\eta, x'')|^p dx'' \leq C \|f\|_{L^p(K)}^p \quad (3.95)$$

Now integrate this inequality over the finite measure space $\mathcal{G}_{k,n}$ to get

$$\begin{aligned} \|Pf\|_{L^p(\mathcal{G}_{k,n})}^p &= \int_{\mathcal{G}_{k,n}} \int_{\eta^\perp} |Pf(\eta, x'')|^p dx'' d\eta \\ &\leq C \|f\|_{L^p(K)}^p \int_{\mathcal{G}_{k,n}} d\eta \\ &= C |\mathcal{G}_{k,n}| \|f\|_{L^p(K)}^p. \end{aligned}$$

This proves that $Pf \in L^p(\mathcal{G}_{k,n})$ and that P restricted to $L^p(K)$ is a bounded operator from $L^p(K)$ to $L^p(\mathcal{G}_{k,n})$.

The remaining part follows from the fact that

$$\begin{aligned} \|Pf\|_{L^p(\mathcal{G}_{k,n})}^p &= \int_{G_{k,n}} \int_{\eta^\perp} |Pf(\eta, x'')|^p dx'' d\eta \\ &= \int_{G_{k,n}} \|Pf\|_{L^p(\eta^\perp)}^p d\eta = \left\| \|P_\eta f\|_{L^p(\eta^\perp)} \right\|_{L^p(G_{k,n})}^p. \end{aligned}$$

■

Remark 3.112. An estimate of the form $\| \|P_\eta f\|_{L^r(\eta^\perp)} \|_{L^q(G_{k,n})} \leq C \|f\|_{L^p(K)}^p$ is called a “mixed norm” estimate. From the theory of normed linear spaces, this is equivalent to showing that the k -plane transform is a continuous, or bounded, linear operator from the Banach space $L^p(K)$ to the Banach space of functions $g(\eta, x'')$ on $\mathcal{G}_{k,n}$ whose “mixed norm”

$$\left\| \|g\|_{L^r(\eta^\perp)} \right\|_{L^q(G_{k,n})} = \left[\int_{G_{k,n}} \left[\int_{\eta^\perp} |g(\eta, x'')|^r dx'' \right]^{\frac{q}{r}} d\eta \right]^{\frac{1}{q}}$$

is finite. Clearly, if $p = q = r$, then $\| \|g\|_{L^r(\eta^\perp)} \|_{L^q(G_{k,n})} = \left[\int_{G_{k,n}} \int_{\eta^\perp} |g(\eta, x'')|^p dx'' d\eta \right]^{\frac{1}{p}}$, and we arrive at the situation in this corollary.

Let us now consider the backprojection operator. We have already established that if g is integrable, then $P^\#g$ is locally integrable (corollary 3.30). We now generalize this to the L^p case.

Proposition 3.113. *If $g \in L^p(\mathcal{G}_{k,n})$, then $P^\#g \in L_{loc}^p(\mathbb{R}^n)$.*

Proof. Let x_0 be an arbitrary point in $\mathbb{R}^n$ and let B be the closed ball of radius 1 centered at x_0 . In the ensuing calculation f represents an arbitrary function in $L^{p'}(B)$. We can then use the formal adjoint theorem to get

$$\left\langle |f|, P^\#|g| \right\rangle_{\mathbb{R}^n} = \langle P|f|, |g| \rangle_{G_{k,n}}$$

From corollary 3.111, $P|f| \in L^{p'}(\mathcal{G}_{k,n})$ and there exists a constant C such that $\|P|f|\|_{L^{p'}(\mathcal{G}_{k,n})} \leq C \|f\|_{L^{p'}(B)}$. By hypothesis g is in $L^p(\mathcal{G}_{k,n})$, so by the previous equation and Hölder's inequality we get

$$\begin{aligned} \left| \left\langle f, P^\#g \right\rangle_{\mathbb{R}^n} \right| &\leq \langle P|f|, |g| \rangle_{G_{k,n}} \\ &\leq \|P|f|\|_{L^{p'}(\mathcal{G}_{k,n})} \|g\|_{L^p(\mathcal{G}_{k,n})} \\ &\leq C \|f\|_{L^{p'}(B)} \|g\|_{L^p(\mathcal{G}_{k,n})} = C' \|f\|_{L^{p'}(B)} \end{aligned}$$

where $C' = C \|g\|_{L^2(\mathcal{G}_{k,n})}$ is constant with respect to $f \in L^{p'}(B)$. This shows that the linear functional λ defined on $L^{p'}(B)$ by $\lambda(f) = \langle f, P^\#g \rangle_{\mathbb{R}^n}$ is continuous. By the

duality between the Banach spaces $L^{p'}(B)$ and $L^p(B)$, there exists some $h \in L^p(B)$ such that

$$\lambda(f) = \langle f, h \rangle_{\mathbb{R}^n}$$

for all $f \in L^{p'}(B)$. Therefore

$$\langle f, P^\# g \rangle_{\mathbb{R}^n} = \langle f, h \rangle_{\mathbb{R}^n}$$

for all $f \in L^{p'}(B)$, so $P^\# g$ restricted to B equals h . Since $h \in L^p(B)$ this shows that the restriction of $P^\# g$ to B is L^p and the proof is complete. ■

The next result is from Solmon [569].

Theorem 3.114. *If $k \geq \frac{n}{p'}$ and $g \in L^p(\mathcal{G}_{k,n})$ is nonnegative and nontrivial, then $P^\# g$ is not in $L^p(\mathbb{R}^n)$.*

Proof. Here we use the result from theorem 3.32 that if $k \geq \frac{n}{p'}$, then there exists a nonnegative function $f \in L^{p'}(\mathbb{R}^n)$ such that $Pf = \infty$ everywhere. By the formal adjoint theorem we then have

$$\infty = \langle Pf, g \rangle = \langle f, P^\# g \rangle$$

But if $P^\# g$ were in $L^p(\mathbb{R}^n)$ then Hölder's inequality would give a finite value for $|\langle f, P^\# g \rangle|$. Hence, the conclusion is established. ■

If we apply theorem 3.114 to the case where $p = 2$, then we see that if $k \geq \frac{n}{2}$ and g is nonnegative, nontrivial, and square integrable on $\mathcal{G}_{k,n}$, then $P^\# g$ is not square integrable on $\mathbb{R}^n$. Because of this result the condition that $P^\# g$ be square integrable and, hence, that g be in the domain of P^* depends entirely on cancellation if $k \geq \frac{n}{2}$. In particular, this is always true for the Radon transform.

3.10.2 The Action of the k -Plane Transform on General L^p Functions

In this section we present the main results and ideas. Most details and proofs are left to the references.

We take advantage of the operator notation for the Riesz potential that we introduced earlier: I^α is the linear operator defined by $I^\alpha f = R_\alpha * f$.

It is possible to invert the Riesz potential of L^p functions for certain values of p without using Fourier analysis. Good references for this approach may be found in Rubin [524] and Samko [541]. We will review some of the results of [524] and [541], but first we show that the Riesz potentials of certain L^p functions exist as functions.

Theorem 3.115. *If $1 \leq p < \frac{n}{\alpha}$, and $0 < \alpha < n$, then $I^\alpha f$ is well defined as a locally integrable function.*

Also, if α is an integer with $1 \leq p < \frac{n}{\alpha}$, and $0 < \alpha < n$, then the α -plane transform of f is well defined.

Proof. It is easy to check that if $1 \leq p < \frac{n}{\alpha}$, then $(1 + |x|)^{\alpha-n} \in L^{p'}(\mathbb{R}^n)$. Then by Hölder's inequality we see that the product of this $L^{p'}$ function with the L^p function f is integrable, that is $(1 + |x|)^{\alpha-n} f(x) \in L^1(\mathbb{R}^n)$. But we proved in theorem 3.64 that this condition implies that $R_\alpha * f$ is well defined as a locally integrable function. The same observation together with theorem 3.35 shows that the k -plane transform exists. ■

With a little more care we could show that theorem 3.115 is true if α is complex and if p satisfies $1 \leq p < \frac{n}{\operatorname{Re} \alpha}$ (compare, Rubin [524]).

If $p \geq \frac{n}{\alpha}$, then it is possible to define the Riesz potential as a certain type of distribution. The test functions are members of the Lizorkin class Φ , which is defined as the space of rapidly decreasing functions all of whose moments vanish. Therefore, $\varphi \in \Phi$ if and only if $\varphi \in \mathcal{S}$ and $\int_{\mathbb{R}^n} x^m \varphi(x) dx = 0$, $m = 0, 1, 2, \dots$. The topological dual space Φ' is the space of Lizorkin class distributions. It can be shown that the Riesz potential is invariant on the Lizorkin space Φ , that is, $I^\alpha(\Phi) = \Phi$ (Samko [541], theorem 2.16). Hence, if $u \in \Phi'$ we can define $I^\alpha(u)$ by $\langle I^\alpha(u), \varphi \rangle = \langle u, I^\alpha(\varphi) \rangle$ and this defines a continuous map $I^\alpha : \Phi' \rightarrow \Phi'$. In general, the Riesz potential of a distribution is not equal to a function, but we do have the following result:

Theorem 3.116 (Samko [541], theorem 7.5). *If $1 \leq p < \infty$ and $0 < \alpha < \infty$, then $I^\alpha(L^p) \subset L_{loc}^p(\mathbb{R}^n)$.*

Hence the Riesz potential of an L^p function is a function that is locally in L^p . If we assume some smoothness conditions on the L^p function, then even more can be said.

Definition 3.117. Let $1 \leq p < \infty$. The Sobolev space W_p^m consists of the subspace of L^p functions all of whose derivatives of order less than or equal to m are in L^p .

Because an L^p function is not generally differentiable, these derivatives are meant in the sense of tempered distributions (see Stein [581] for more details on this type of Sobolev space).

Theorem 3.118. *If $1 < p < \infty$ and if m is a nonnegative integer, then $W_p^m = I^m(L^p) \cap L^p$. More generally, $W_p^m \subset I^\alpha(L^p) \cap L^p$ if $1 < p < \infty$ and m is a real number with $m > \alpha$.*

Therefore, every Sobolev space function in W_p^m is the Riesz potential of an L^p function. This theorem is proved by Samko [541] (corollary to theorem 7.9; see also, [541], theorem 7.30).

We now consider methods of finding a left inverse for the Riesz potential of certain L^p functions. This is interesting for us because of backprojection theorem 3.67, which in the terminology of this section states that $P^\# P f = c \cdot I_k f$. Therefore, the existence of a left inverse for the Riesz potential implies an inversion formula for the k -plane transform.

We actually know how to do this for functions f such that

$$f \in L^2, (1 + |x|)^{\alpha-n} f(x) \in L^1 \text{ and } |\xi|^{-\alpha} \widehat{f}(\xi) \in L_{loc}^1. \quad (3.96)$$

By the Riesz multiplier theorem 3.74 Λ^α is a left inverse of the Riesz potential I^α for such functions.

When the Riesz potential I^α has a left inverse, then this inverse is called the **Riesz fractional derivative** and is denoted by $\mathbb{D}^\alpha$.

Therefore, $\mathbb{D}^\alpha$ and Λ^α agree on $I^\alpha f$ for functions satisfying condition (3.96). Rubin [524] and Samko [541] have methods for showing that the Riesz fractional derivative exists for the Riesz potentials of certain L^p functions. For example, both Rubin [524] and Samko [541] prove: if $f \in L^p$ with $1 \leq p < \frac{n}{\operatorname{Re} \alpha}$, then the Riesz fractional derivative exists for $I^\alpha f$, and hence, $\mathbb{D}^\alpha I^\alpha f = f$ ([524], pp. 238–239; [541], theorem 3.19).

However, Rubin [524] has a more general result that we will describe in some detail. We refer the reader to [524] for the proofs. Let P be the Poisson kernel⁶: $P(x, t) = c_n \frac{t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}$, defined on $\mathbb{R}^n \times \mathbb{R}$, where $c_n = \frac{1}{2}|S^n|$. Define

$$\begin{aligned} \varkappa_m(\alpha) &= \int_0^\infty (1 - e^{-t})^m \frac{1}{t^{\alpha+1}} dt \quad \text{and} \\ \mathbb{D}_{\varepsilon, m}^\alpha f(x) &= \frac{1}{\varkappa_m(\alpha)} \int_0^\infty \sum_{j=0}^m (-1)^j \binom{m}{j} P * f(x, jt) \frac{1}{t^{\alpha+1}} dt. \end{aligned}$$

Rubin is able to show that if m satisfies $m > \alpha$, then $\varkappa_m(\alpha) \neq 0$. Furthermore, if $f \in L^p$ and $I^\alpha f \in L^r + L^s$ for some $p \geq 1$ and some r and s with $1 \leq r, s < \infty$, then $\lim_{\varepsilon \rightarrow 0^+} \mathbb{D}_{\varepsilon, m}^\alpha(I^\alpha(f))$ converges in the L^p norm and is independent of the choice of m as long as $m > \alpha$. This means that the operator $\mathbb{D}^\alpha$ is well defined at the function $I^\alpha(f)$ by the equation $\mathbb{D}^\alpha(I^\alpha(f)) = \lim_{\varepsilon \rightarrow 0^+} \mathbb{D}_{\varepsilon, m}^\alpha(I^\alpha(f))$. Furthermore, Rubin shows that $\mathbb{D}^\alpha$ is the Riesz fractional derivative. More precisely, we have the following theorem:

Theorem 3.119 (Rubin [524], theorem 16.4). *If $f \in L^p$ and $I^\alpha f \in L^r + L^s$, $1 \leq p < \infty$, $1 \leq r, s < \infty$, then the Riesz fractional derivative $\mathbb{D}^\alpha$ exists and we have:*

$$\mathbb{D}^\alpha I^\alpha f = f.$$

Remark 3.120. Where are we to find functions with $I^\alpha f \in L^r + L^s$. Theorem 3.118 gives the answer. We can find many such functions in appropriate Sobolev spaces.

An immediate consequence of theorem 3.119 is the following inversion formula for the k -plane transform of certain L^p functions. It is a direct consequence of Rubin's theorem on the Riesz potential, compare, theorem 3.119. Rubin [528] has more detailed results.

Theorem 3.121 (Inversion of the k -plane transform: Rubin). *Let $f \in L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$, $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, and $I^k f \in L^r + L^s$ for some r and s*

⁶ We use P to denote the Poisson kernel only in this paragraph. All other instances of P denote the k -plane transform.

with $1 \leq r, s < \infty$. Also let P be the k -plane transform on $\mathbb{R}^n$. Then

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \mathbb{D}^k P^\# P f = f$$

Proof. Once we know that the Riesz fractional derivative is a left inverse of the Riesz potential, then the same method of proof we used in theorem 3.79 works to give an inversion theorem for the k -plane transform. Basically we observe that the backprojection theorem exhibits the Riesz potential as $R_k * f(x) = \frac{1}{(2\pi)^k |G_{k,n-1}|} P^\# P f(x)$. An application of the Riesz fractional derivative then completes the derivation of the inversion formula. The conclusion of the theorem is thus a consequence of Rubin's theorem 16.4 [524], which shows that the Riesz fractional derivative is a left inverse for the Riesz potential. ■

There are some interesting consequences to this inversion theorem, but to prove them we need part of the Hardy–Littlewood–Sobolev theorem on fractional integration, which we state without proof. The complete statement and the proof may be found in Stein [581], page 119.

Theorem 3.122 (Hardy–Littlewood–Sobolev). *Given a real number α with $0 < \alpha < n$ and given a real number p with $1 < p < \frac{n}{\alpha}$, then for every $f \in L^p(\mathbb{R}^n)$, the integral defining $I^\alpha f$ is absolutely convergent. Furthermore if we define q so that*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

then there is a constant $A_{p,q}$ such that $\|I^\alpha f\|_{L^q} \leq A_{p,q} \|f\|_{L^p}$. In particular, the Riesz potential maps L^p into L^q .

The next two results are according to Rubin.

Corollary 3.123 (Rubin [528, 538]). *Let $f \in L^p(\mathbb{R}^n)$ for $1 < p < \frac{n}{k}$, and let P be the k -plane transform on $\mathbb{R}^n$. Then*

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \mathbb{D}^k P^\# P f = f$$

Proof. The hypothesis allows us to apply theorem 3.115 to show that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. It also allows us to apply the Hardy–Littlewood–Sobolev theorem to show that there is some $q > 1$ with $I^k f \in L^q$. All the hypotheses of theorem 3.121 are satisfied, so the conclusion is valid. ■

The following result is originally from Jensen [321] who required a weak Hölder condition on the function f . Rubin [538] was able to derive Jensen's result without the weak Hölder condition. We present just a slightly more general version that includes the results of both [321] and [538].

Corollary 3.124 (Jensen–Rubin). *Let $0 < k < n$ and let P be the k -plane transform on $\mathbb{R}^n$. If f is a measurable function which is, almost everywhere, $O((1 + |x|)^{-N})$ for*

some $N > k$, then we have the inversion formula

$$\frac{1}{(2\pi)^k |G_{k,n-1}|} \mathbb{D}^k P^\# P f = f$$

This condition, and hence the conclusion, is satisfied if f is continuous and $f = O(|x|^{-N})$.

Proof. First note that the hypothesis implies $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ so the k -plane transform is well defined.

Since $N > k$ we can choose a real number p satisfying $\frac{n}{N} < p < \frac{n}{k}$. Also since $n > k$, the choice of p may be made so that $p > 1$.

By the hypothesis, there is a constant $C > 0$ such that $|f(x)| \leq C(1 + |x|)^{-N}$ for almost all x . Let A be the set where $|x| \leq 1$ and let B be the set where $|x| > 1$. Let f_A be the restriction of f to A and let f_B be the restriction of f to B .

On A we have $|f_A(x)| \leq C(1 + |x|)^{-N} \leq C$ almost everywhere. Since f_A is measurable and bounded on a compact set, then $f_A \in L^p$.

On B we have $|f_B(x)| \leq C(1 + |x|)^{-N} \leq C|x|^{-N}$ almost everywhere. Since $n < Np$ it follows that $|x|^{-N}$ restricted to B , and hence also $f_B(x)$, is L^p . Therefore, f is the sum of two L^p functions. Since $1 < p < \frac{n}{k}$, we can apply corollary 3.123 to obtain the desired conclusion. ■

Remark 3.125. The books by Samko [541], and Rubin [524] are instructive beyond merely showing the existence of the left inverse of the Riesz potential operator. They also show how to get good approximations to the inverses in terms of Marchaud-type integrals defined by finite differences [524, 541]. Rubin [524] also shows how to use continuous wavelets to obtain the left inverses.

See Rubin [525, 526, 528, 537, 538] for more applications of these ideas to Radon transforms and k -plane transforms.

Granting that Rubin’s results on the inversion of the k -plane transform for L^p functions are more general than the inversion theorem 3.79, which we developed earlier, it is interesting to compare both approaches in the case $p = 2$. In this case we saw that both approaches yielded inversion formulas for the k -plane transform of any L^2 function, provided $1 < k < \frac{n}{2}$ (theorem 3.86 and corollary 3.123). For big “Oh” functions, however, the Jensen–Rubin corollary has a much better result than our corollary 3.82.

For the k -plane transform with $k \geq \frac{n}{2}$ both approaches require $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$. Our approach requires the auxiliary condition $|\xi|^{-k} \widehat{f}(x) \in L^1_{\text{loc}}$, whereas Rubin’s approach requires the auxiliary condition $I^k f \in L^r + L^s$. Using our approach we were able to give a very simple proof that if $0 < k < n$, $2 \leq p < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then the inversion formula holds (corollary 3.81). We now show that this result is obtainable by Rubin’s approach. The proof is somewhat less elementary because it requires the Hardy–Littlewood–Sobolev theorem on fractional integration. On the other hand we get a more general result than corollary 3.81.

Corollary 3.126. *If $0 < k < n$, $1 < p < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, then*

$$f = \frac{1}{(2\pi)^k |G_{k,n-1}|} \mathbb{D}^k P^\# P f.$$

Proof. If $1 < p < \frac{n}{k}$, then the result is immediate from corollary 3.123.

The case remains where $p \geq \frac{n}{k}$. In this case we wish to use theorem 3.121. The hypothesis that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$ follows from the current hypothesis that $f \in L^1(\mathbb{R}^n)$. Because f is also in $L^p(\mathbb{R}^n)$, then it suffices to show that $I^k f \in L^r$, for some $r \geq 1$. Choose a real number m with $1 < m < \frac{n}{k}$. In particular we have $m < p$. Let $A = \{x \in \mathbb{R}^n : |f(x)| \geq 1\}$, let $B = \{x \in \mathbb{R}^n : |f(x)| < 1\}$ and define $f_1 = f|_A$, $f_2 = f|_B$. Then, since $m < p$, we have $|f(x)|^m \leq |f(x)|^p$ on A . Since $f \in L^p$, it follows that $f_1 \in L^m$. Also $|f(x)|^m < |f(x)|$ on B , so since $f \in L^1$, then $f_2 \in L^m$. This implies that $f \in L^m(\mathbb{R}^n)$ since $f = f_1 + f_2$.

Since $1 < m < \frac{n}{k}$ we can use the Hardy–Littlewood–Sobolev theorem on fractional integration to obtain some $r > 1$ with $\|I^k f\|_{L^r} \leq c \cdot \|f\|_{L^m} < \infty$. This shows that $I^k f \in L^r$. We have shown that all the hypotheses of theorem 3.121 apply in this case, so we have the desired inversion formula. ■

Note that this corollary gives a nice counterpoint to corollary 3.123.

Both corollary 3.81 and corollary 3.126 show that the k -plane transform is invertible on the classes of compactly supported L^p functions, $2 \leq p < \infty$, and Schwartz class functions, which are the most common functions used in applications of tomography.

It would be interesting to generalize theorem 3.79 to the L^p case. However, if $p > 2$, then there is no guarantee that $\hat{f}$ is a function, although it is a tempered distribution. But if it is a distribution of an order greater than 0, then there is no way of making sense of the product $|\xi|^{-k} \hat{f}(x)$, in general. Even if $1 \leq p < 2$, there is no obvious way of generalizing our proof of theorem 3.79 to the L^p case.

Mixed Norm Estimates for the k -Plane Transform

We next investigate mixed norm estimates for the k -plane transform (compare, remark 3.112).

Definition 3.127. We define the q, r *mixed-norm* of a function g defined on $\mathcal{G}_{k,n}$ by

$$\|g(\eta, x'')\|_{L^q(L^r)} = \left[\int_{\eta \in G_{k,n}} \left\{ \int_{x'' \in \eta^\perp} |g(\eta, x'')|^r dx'' \right\}^{q/r} d\eta \right]^{1/q}.$$

The interest in mixed-norm estimates began with Solmon's 1979 paper [570]. As we know from earlier in this book, Solmon's paper derived many fundamental results about the k -plane transform. In this paper, Solmon gave some mixed-norm estimates for the k -plane transform of L^p functions, $p \leq 2$, $p < n/k$, and this was the original idea in developing mixed-norm estimates of the type in theorem 3.128 which we present below.

Strichartz [584] gave some L^p estimates ($1 < p \leq 2$) for the k -plane transform on $\mathbb{R}^n$. He used these estimates to show that $Pf(\eta, x'')$ has $\frac{k}{p'}$ derivatives in the variable x'' , where p' is the conjugate index to p . He then went on to define and study Radon transforms on hyperbolic and real projective spaces. Peters [476, 478] also has some results in this direction.

The best result for mixed-norm estimates for the k -plane transform is the following theorem of M. Christ. Refer to Christ [91] for the proof.

Theorem 3.128 (Christ [91]). *If P is the k -plane transform on $\mathbb{R}^n$, and k, p, q, r , and n are related by*

$$\begin{aligned} 1 \leq p &\leq \frac{n+1}{k+1} \\ np^{-1} - (n-k)r^{-1} &= k \\ q &\leq (n-k)p' \end{aligned}$$

then P is a bounded linear operator $P : L^p(\mathbb{R}^n) \rightarrow L^q(\mathcal{G}_{k,n})$, where $\mathcal{G}_{k,n}$ is provided with the q, r mixed norm and we have the estimate

$$\|Pf(\eta, x'')\|_{L^q(L^r)}^q \leq C \|f\|_{L^p}$$

The same result is true if the condition $1 \leq p \leq \frac{n+1}{k+1}$ is replaced by the conditions $1 \leq p < \frac{n}{k}, 1 \leq p \leq 2$.

Christ's theorem is a generalization to the k -plane transform of the estimates of Oberlin and Stein [455] who proved Christ's results for the Radon transform with $1 \leq p < \frac{n}{n-1}, q \leq p'$, and $\frac{1}{r} = \frac{n}{p'}$.

It can be shown that the first two conditions $np^{-1} - (n-k)r^{-1} = k$ and $q \leq (n-k)p'$ are necessary if the mixed norm estimate is to be true. Drury [132] showed that $P : L^p(\mathbb{R}^n) \rightarrow L^q(\mathcal{G}_{k,n})$ is bounded provided that $n \leq 2k+1, 1 \leq q \leq n+1$, and $np^{-1} - (n-k)q^{-1} = k$. He also conjectured that this would be true with the two necessary conditions and the condition $1 \leq p < \frac{n}{k}$. Therefore, Christ resolved Drury's conjecture in the case $r = q$ except for p in the range $\frac{n+1}{k+1} < p < \frac{n}{k}$. Drury [132] is a generalization of an earlier paper, Drury [131], in which the case $k = 1$, the x-ray transform, was handled (compare, Calderón [74], theorem 2).

Drury [135] has a very nice survey of mixed-norm estimates for the k -plane transform as of 1989. This paper includes some basic examples and sketches of the main techniques.

If we restrict the set of functions, then more precise results are possible; for example, Duoandikoetxea, Naibo, and Oruetebarria [137] have the following version of Christ's theorem for radial functions:

Theorem 3.129 (Duoandikoetxea et. al. [137]). *Let P be the k -plane transform on $\mathbb{R}^n$. Then the mixed-norm estimate*

$$\|Pf(\eta, x'')\|_{L^q(L^r)}^q \leq C \|f\|_{L^p}$$

is true for all radial functions if and only if k, p, q, r , and n are related by

$$\begin{aligned} 1 &\leq p < \frac{n}{k} \\ np^{-1} - (n - k)q^{-1} &= k \\ 1 &\leq r \leq \infty \end{aligned}$$

In this paper the authors also have a “reversed” mixed-norm estimate for radial functions. They also have a discussion of how the construction of a higher dimensional Besicovitch type set would lead to boundary information on the values of k and q in the mixed-norm estimates for the k -plane transform.

Duoandikoetxea [136] has the following result for the x-ray transform:

Theorem 3.130 (Duoandikoetxea [136]). *Let P be the k -plane transform with $k = 1$ on $\mathbb{R}^n$, i.e. we are dealing with the x-ray transform. Then, if $n = 2$, the mixed-norm estimate*

$$\|Pf(\eta, x'')\|_{L^q(L^r)}^q \leq C\|f\|_{L^p}$$

is true for all L^p functions if and only

$$\begin{aligned} 1 &\leq p < 2 \\ np^{-1} - (n - 1)q^{-1} &= 1 \\ r &\leq 2(n - 1). \end{aligned}$$

If $n \geq 3$, then the mixed-norm estimate

$$\|Pf(\eta, x'')\|_{L^q(L^r)}^q \leq C\|f\|_{L^p}$$

is true for all L^p functions if and only

$$\begin{aligned} 1 &\leq p < \frac{n+1}{2} \\ np^{-1} - (n - 1)q^{-1} &= 1 \\ r &\leq p'(n - 1). \end{aligned}$$

Duoandikoetxea [136] also contains analogous mixed-norm results for potential type operators related to the Riesz potential.

3.11 Local Tomography

We say that an algorithm involving a Radon transform R to reconstruct an operator quantity Q is **local** if reconstructing the value $Qf(x)$ requires only the knowledge of the Radon transform of f for hyperplanes passing close to x . An example of a local algorithm is the formula for inverting the Radon transform in odd dimensions

$$f(x) = \frac{(-1)^{\frac{n-1}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

(see theorem 3.53). It is clear from this formula that only hyperplanes containing x are needed to reconstruct $f(x)$. But if n is even then the corresponding formula is

$$f(x) = \frac{(-1)^{\frac{n-2}{2}}}{2^n \pi^{n-1}} \int_{S^{n-1}} H \frac{\partial^{n-1} Rf}{\partial p^{n-1}}(\theta, \langle x, \theta \rangle) d\theta$$

and the presence of the Hilbert transform demands values of $Rf(\theta, y)$ for all real y .

Local tomography is concerned with imaging a, possibly small, region of interest in a larger object. In the example of CT a local algorithm will reduce the time and radiation exposure necessary to reconstruct the region of interest because one need only send radiation through a neighborhood of the region of interest. In odd dimensions this can be done directly with the Radon transform. However, in the important case $n = 2$ this is not possible with the usual algorithms.

Local tomography was originally defined as the process of reconstructing Λf rather than f from the Radon transform of f .⁷ This form of tomography is local in even dimensions and gives other useful information, especially relating to boundaries between components of an object. However, this form of tomography distorts the interior of component objects. More recent developments have provided local algorithms to reconstruct f rather than Λf . For this reason the reconstruction of Λf is now called **lambda tomography** and lambda tomography is considered to be a subset of local tomography.

Lambda tomography was first studied by Smith and Keinert [564] and Vainberg, Kazak, and Kurozaev [607].

What is the interest in reconstructing Λf ? This is explained very well by the following quote from the 1992 paper of Faridani, Ritman, and Smith [163] page 460: “During the past year, local tomography has become the standard procedure at the Biodynamics Research Unit of the Mayo Clinic for defining the anatomic outline of the lungs and for the study of the coronary arterial tree. With regular tomography, the lung edge is sufficiently blurred that no consistent edge is detected, while consistent edges are detected with the sharp boundaries of local tomography.”

Therefore lambda tomography is important when it is desired to see structural differences, whereas standard tomography would be of more interest when it is desired to know the precise density of the tissue or material being examined.

Lambda tomography also offers a reduction in radiation exposure and greater computational efficiency as compared with standard tomography in even dimensions.

Before proceeding, let us say that we will be informal and not worry about regularity of functions and the various hypotheses of the theorems we use. More rigorous presentations can be found in Faridani et al. [160, 161], Faridani, Ritman, and Smith [162, 163], Smith and Keinert [564], and Keinert [348].

First, let us investigate why lambda tomography emphasizes boundaries. Assume that we have an object represented by the function f and that this object is composed of disjoint parts with fairly constant density. If we move across a boundary of objects with

⁷ Recall definition 3.39 of the lambda operator.

differing densities, then $\widehat{f}$ the Fourier transform of f tends to be larger in the region of the boundary, since it requires higher frequency components to transition across the boundary. Since $(\Lambda f)^\wedge(\xi) = |\xi| \widehat{f}(\xi)$, the higher frequency components of f tend to be exaggerated in Λf as compared with f and hence emphasize boundaries.

By the extended inversion theorem 3.106 with $k = n - 1, s = n$ we have

$$\frac{1}{(2\pi)^{n-1}} \Lambda^{-1} R^\# \Lambda^n R f = f$$

Applying the lambda operator to both sides yields

$$\Lambda f = \frac{1}{(2\pi)^{n-1}} R^\# \Lambda^n R f \quad (3.97)$$

and this gives a method of reconstructing Λf from Rf .

If we now assume that n is even, then by the derivative theorem for the Fourier transform

$$\mathcal{F}(\Lambda^n R f)(\theta, q) = q^n \widehat{R f}(\theta, q) = \mathcal{F}\left(\frac{d^n}{dp^n} R f\right)(\theta, q)$$

and hence $\Lambda^n R f = \frac{d^n}{dp^n} R f$. Note that we were able to replace $|q|^n$ by q^n since n was even and that was crucial in obtaining the derivative. Using this result in equation (3.97) and expanding the backprojection operator gives

$$\Lambda f(x) = \frac{1}{(2\pi)^{n-1}} \int_{S^{n-1}} \frac{d^n}{dp^n} (R f)(\theta, \langle x, \theta \rangle) d\theta$$

and from this we see that this algorithm for reconstructing Λf is local in even dimensions in the same way that we determined that the usual Radon transform is local in odd dimensions.

Berenstein and Walnut [50, 51] use continuous wavelet transforms to derive an approximate inversion formula to obtain f from its Radon transform Rf . Their algorithm is local in even dimensions. Because it reconstructs f instead of Λf , the distortions induced by lambda tomography are avoided. If one wants to reconstruct $f(x)$ to within an error ε , Berenstein and Walnut show that it is sufficient to know Rf on hyperplanes which intersect a ball centered at x . The radius of the ball depends on ε . Also see Berenstein [42] for a short overview of these ideas and some very interesting pictures of local reconstructions.

Katsevich [334, 336, 338] studies lambda tomography for various generalized Radon transforms including the attenuated and exponential Radon transforms (see chapter 5 for a definition of these transforms).

Katsevich and Ramm [346] study lambda tomography, pseudolocal tomography, and geometric tomography for finding the singular support of a function.

Kuchment, Lancaster, and Mogilevskaya [361] investigate some aspects of local tomography for some generalized Radon transforms, including the exponential Radon transform.

Another local method is pseudolocal tomography that was introduced by Katsevich and Ramm [347]. Given the data from the Radon transform of a function f on $\mathbb{R}^n$, pseudolocal tomography provides an algorithm to reconstruct a function f_d for every $d > 0$ with the property that $f - f_d$ is continuous. This allows one to find all discontinuities of f from f_d . The advantage of f_d is that to detect a discontinuity at x one only needs the Radon transform for hyperplanes whose distance from x is less than or equal to d . Katsevich [333] shows how to do pseudolocal tomography for the exponential Radon transform.

In addition to the references already cited, we should mention that Faridani [157] has a brief, but excellent introduction to lambda tomography and Faridani et al. [159] have a very nice introductory survey of all the areas of local tomography. Ramm and Katsevich [513] is a booklength treatment of their contributions to local tomography. Their book also has an introduction to the Radon transform and ordinary tomography. Also of interest are Ramm [511, 512], Berenstein [41], Katsevich [337, 342], and Madych [412].

3.12 Uniqueness and Nonuniqueness: The Kernel of the Radon Transform

The x-ray transform, X , is the k -plane transform for $k = 1$. If f represents an object in the plane, then $Xf(\eta, x'')$ represents the total density of the object along the line defined $\eta + x''$. This models the information that could be obtained by sending an x-ray along this line (recall the discussion in chapter 1 concerning the relationship between x-rays and the Radon transform).

Therefore, in the higher-dimensional case of the k -plane transform for $k > 1$, we will think of $P(\eta, x'')$ as a “generalized (k -dimensional) x-ray.” For simplicity, we will refer to these as “x-rays.” We also use the term “radiographs.”

It is natural to ask how many x-rays are necessary to uniquely determine an object. A related question is: what is the size of the kernel of the k -plane transform? For example, if the kernel is trivial, then a function is uniquely determined by all its generalized x-rays.

We recall that the k -plane transform is a linear transformation and that it is well defined on both L^1 and on the space of measurable functions f such that $(1 + |x|)^{k-n} f(x) \in L^1$. In this section k is an integer with $0 < k < n$, and P denotes the k -plane transform.

The first result is a direct consequence of the slice-projection theorem. It is an old and well-known result having appeared, for $k = n - 1$, in Besicovitch [53], Cramér and Wold [110], Green [230], and Newman [447].

Theorem 3.131. *The kernel of the k -plane transform acting on $L^1(\mathbb{R}^n)$ is trivial. Hence $P : L^1(\mathbb{R}^n) \rightarrow L^1(\mathcal{G}_{k,n})$ is an injective bounded linear transformation.*

Proof. If $f \in L^1$ with $P(f) = 0$, then also $\widehat{Pf} = 0$. Then by the slice-projection theorem (theorem 3.27), for all $\eta \in G_{k,n}$ we have $\widehat{f}(\xi'') = (2\pi)^{-\frac{k}{2}} \widehat{Pf}(\eta, \xi'') = 0$ for all ξ'' in the $n - k$ plane $\eta^\perp$. But, as η runs through all the k planes through the origin, the union of all the $n - k$ planes $\eta^\perp$ is equal to $\mathbb{R}^n$. Hence, the Fourier transform of f , and thereby f itself, vanishes.

This proves that the operator $P : L^1(\mathbb{R}^n) \rightarrow L^1(\mathcal{G}_{k,n})$ is injective. The statement about the k -plane transform being a bounded linear operator was proved in theorem 3.25. ■

Corollary 3.132. *If f and g are L^1 functions with the same k -plane transform, then they are identical.*

The same type of result holds on any linear space of functions for which we have an inversion theorem. In particular, by theorem 3.79 we have:

Theorem 3.133. *The kernel of the k -plane transform is trivial on the space of L^2 functions f such that $(1 + |x|)^{k-n} f(x) \in L^1$ and $|\xi|^{-k} \widehat{f}(\xi)$ is locally integrable.*

Hence, if f and g are functions in this space and they have the same k -plane transform, then they are identical.

For these uniqueness theorems it is important that the functions belong to a space such as L^1 on which the growth of the functions is limited in some way. Zalcman [626] has an example of a nonzero smooth function on $\mathbb{R}^2$ whose Radon transform vanishes on all lines. Armitage and Goldstein [27] also have such an example on $\mathbb{R}^n$ (compare, Armitage [26]). Precisely we have

Theorem 3.134 (Zalcman nonuniqueness theorem [626]). *There is a nontrivial harmonic function h defined on $\mathbb{R}^2$, such that the Radon transform of h vanishes on every line.*

Theorem 3.135 (Armitage-Goldstein nonuniqueness theorem [27]). *There is a nontrivial harmonic function h defined on $\mathbb{R}^n$, such that the Radon transform of h vanishes on every hyperplane.*

The Zalcman nonuniqueness theorem is proved by using a theorem of Arakelian [20] to construct an entire function g on the plane such that for every line l in the plane

$$\begin{aligned} g' &\in L^1(l), \\ g(x) &\rightarrow 0 \quad \text{as } |x| \rightarrow \infty \text{ and} \\ g &\text{ is not identically zero} \end{aligned} \tag{3.98}$$

Then if we define $h = g'$, then h is integrable on every line. Each line l can be parametrized to be in one-to-one correspondence with the real line $\mathbb{R}$. We abuse the language by letting a represent both a real number and the corresponding point on the line l . Because h is integrable on every line l , we can use the Lebesgue dominated convergence theorem and the fundamental theorem of calculus to obtain

$$\begin{aligned} \int_l h(x) dx &= \lim_{a \rightarrow \infty} \int_{-a}^a g'(x) dx \\ &= \lim_{a \rightarrow \infty} (g(a) - g(-a)) = 0 \\ &\quad \text{by (3.98)} \end{aligned}$$

However, h is not identically zero. For if it were, then g would be constant and the condition $g(x) \rightarrow 0$ as $|x| \rightarrow \infty$ would imply that g were identically zero and this would contradict the construction of g .

We refer the reader to the cited papers for more details.

Remark 3.136. The integrals defining the Radon transform of h exist and are finite on all hyperplanes of $\mathbb{R}^n$. However, h itself is not integrable on $\mathbb{R}^n$, as theorem 3.131 shows.

Remark 3.137. Helgason [291], theorem 6.2, states that the k -plane transform is invertible on the space of C^∞ functions f satisfying the growth condition $f(x) = O(|x|^{-N})$ for $N > n$. Some such growth condition is necessary for inversion or uniqueness theorems, because the nonuniqueness results of Zalcman and of Armitage and Goldstein show that, in general, the kernel of the k -plane transform is nontrivial.

In Zalcman's proof it is shown that h satisfies the growth condition $O(|x|^{-2})$ on every line in the plane. However, Helgason's result shows that this growth condition must fail if we let x range through the entire plane.

Recently Jensen [321] proved the much more general result that the k -plane transform is invertible on the space of functions which are $O(|x|^{-N})$ for some $N > k$ and which satisfy a weak Hölder condition, compare Rubin [538] and remark 3.83 and corollary 3.124.

We have therefore shown that generalized x-rays in the guise of k -plane transform projections P_η completely determine an L^1 function if these x-rays are known for all k spaces η . But the Zalcman–Armitage–Goldstein results show that this may not be true if we do not restrict the growth of the function in some way.

It is interesting to see if we can get uniqueness results that do not require the projections for all k spaces. More informally we can ask whether we need x-rays from all directions to determine an object or whether some restricted set of directions would do.

In their paper [566], Smith, Solmon, and Wagner make the following assertions:

- An object is determined by any infinite set of radiographs.
- For almost any finite dimensional space of objects F , the objects in F can be distinguished by a single radiograph from almost any direction.
- In general, a finite set of radiographs tells nothing at all.

Although these assertions may seem to be improbable and perhaps contradictory, their meanings may be understood by examining their precise statements. The precise statements will resolve any apparent contradictions. Theorems 3.138, 3.144, and 3.148, which are proved below, address these issues. Theorem 3.144 proves the result about an infinite set of radiographs, and theorem 3.148 proves the second assertion about a single radiograph. The third assertion is made precise in theorem 3.138, which we deal with first.

Informally, the third assertion means that if we are given a compactly supported function f and finitely many directions $\theta_1, \dots, \theta_m$, then we can find a compactly

supported function g with identically the same x-ray projections from these directions, yet which differs as much as one might desire on most of the support set. For example, an object that we are imaging, say a brain, may have exactly the same x-ray projections from a given finite number of directions as a picture of the Mona Lisa, except that the second picture will differ from the actual Mona Lisa on a small ring near the boundary of the support set. Of course some really bad things are happening in this small ring for the x-rays to be the same. That is to say, inside this small ring are very large oscillations which allow for the images to be so different yet have the same specified x-rays. Because most practical medical-imaging algorithms use a low-bandpass filter, then there is little practical import to this curious result whose precise statement is:

Theorem 3.138 (Smith, Solmon, Wagner nonuniqueness theorem). *Given a finite number of directions $\theta_1, \dots, \theta_m$ in $\mathbb{R}^n$, a compactly supported C^∞ function f and a compact subset K of $\text{supp}(f)$, then there is a C^∞ function g supported in $\text{supp}(f)$ such that g is completely arbitrary on K yet $X_{\theta_j} f = X_{\theta_j} g$ for $j = 1, \dots, m$.*

This theorem follows as a special case of theorem 3.146 which is stated and proved later.

Here is a graphic illustration of this theorem in $\mathbb{R}^2$. Because the *x-ray* transform is essentially the Radon transform in $\mathbb{R}^2$, we use $R_\theta f$ in place of $X_\theta f$. In the accompanying figure, the function $f \in L_0^2(\mathbb{R}^2)$ represents a picture of the Mona Lisa, which is supported in the unit disk, whereas g represents a picture of an F-16 jet flying over a pyramid. Let K be the compact set which is the complement, relative to the unit disk, of the set A represented by the thin annulus on the boundary of the supporting disk. We could agree that the function g is arbitrarily different than the function f on the set K . By theorem 3.138 it is quite possible to modify the picture of the jet on the set A so that $R_\theta f = R_\theta g$ for 100,000 directions θ . By suitable modifications inside this annulus we could even have equality of the x-ray projections for a million, a trillion, or indeed for any finite number of directions. Nonetheless, the pictures are dramatically different from each other at all points.

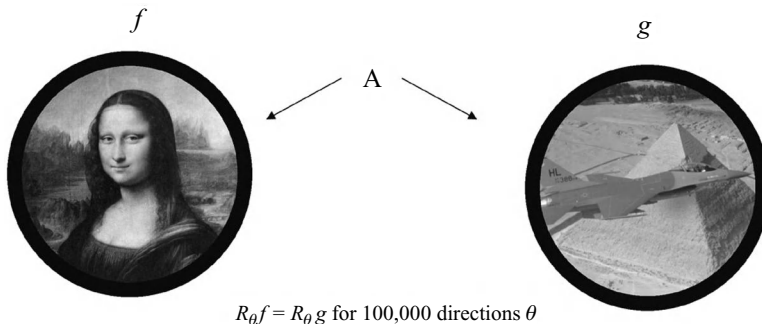


Image on right courtesy of the United States Air Force.

It is in this sense that a finite set of radiographs tells nothing at all about a given object.

We proceed by proving a generalization of the first assertion of Smith, Solmon, and Wagner that an object is determined by any infinite set of radiographs. For this we need a few preliminary results.

Recall that L_0^1 denotes the set of compactly supported integrable functions on $\mathbb{R}^n$. The following result is a special case of the Paley–Wiener theorem.

Lemma 3.139. *If $f \in L_0^1$, then its Fourier transform $\widehat{f}$ is the restriction to $\mathbb{R}^n$ of an entire analytic function on $\mathbb{C}^n$.*

Proof. Define

$$F(z) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x, z \rangle} dx.$$

Since f is compactly supported, the integrand is Lebesgue integrable for any fixed complex value of z , despite the fact that $|e^{-i\langle x, z \rangle}|$ may no longer be equal to 1 and, in fact, may get quite large.

Now consider a partial differential operator L with constant coefficients applied to $F(z)$. Because of the considerations of the first paragraph, this operator can be taken under the integral sign, thus yielding

$$L(F(z)) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) L(e^{-i\langle x, z \rangle}) dx \quad (3.99)$$

If we now let $L = \frac{\partial}{\partial z_j}$ be the Cauchy–Riemann operator, the analyticity of $e^{-i\langle x, z \rangle}$ shows that $L(e^{-i\langle x, z \rangle}) = 0$ for all x , so by equation (3.99) $LF = 0$. Since this is true for all $j = 1, \dots, n$, then by the Cauchy–Riemann theorem in several complex variables it follows that F is an entire analytic function. Since it is obvious that $\widehat{f}(\xi) = F(\xi)$ for real valued ξ , the proof is finished. ■

Lemma 3.140. *If Q is a homogeneous polynomial of degree N defined on $\mathbb{R}^n$ which vanishes on $N + 1$ distinct hyperplanes through the origin, then Q is identically zero.*

Proof. Let the given hyperplanes through the origin be defined by the equations $\langle x, \theta_j \rangle = 0$, $j = 1, \dots, N + 1$, where each $\theta_j \in S^{n-1}$. Each such hyperplane locally has measure zero and, hence, so does the union of these hyperplanes. Therefore there must be at least one unit vector α which is not in any of these hyperplanes. This means that $\langle \alpha, \theta_j \rangle \neq 0$ for $j = 1, \dots, N + 1$. Now consider the linear functional $\lambda(x) = \frac{\langle x, \theta_j \rangle}{\langle \alpha, \theta_j \rangle} - \frac{\langle x, \theta_m \rangle}{\langle \alpha, \theta_m \rangle}$ for $j \neq m$. Because the hyperplanes orthogonal to θ_j and θ_m are distinct, there is a point x_0 in the second hyperplane but not the first. This means that $\lambda(x_0) = \frac{\langle x_0, \theta_j \rangle}{\langle \alpha, \theta_j \rangle} \neq 0$. Thus, λ is a linear functional on $\mathbb{R}^n$ which is not identically zero. Therefore, its kernel is a hyperplane through the origin and hence, locally, has measure zero. The union of all these kernels as $j \neq m$ ranges through $1, \dots, N + 1$ also is locally of measure zero and hence the complement, which we denote by S must be dense in $\mathbb{R}^n$.

Let x_1 be any point in S . Then we have

$$\frac{\langle x_1, \theta_j \rangle}{\langle \alpha, \theta_j \rangle} \neq \frac{\langle x_1, \theta_m \rangle}{\langle \alpha, \theta_m \rangle}.$$

for all $j \neq m$. This shows that if we define $t_j = -\frac{\langle x_1, \theta_j \rangle}{\langle \alpha, \theta_j \rangle}$, then $t_1, \dots, t_{N+1}$ are distinct real numbers. An easy calculation shows that $x_1 + t\alpha_j \perp \theta_j$ so $Q(x_1 + t_j\alpha) = 0$ since Q vanishes on the hyperplane orthogonal to θ_j .

Now consider the polynomial in the single real variable t defined by $p(t) = Q(x_1 + t\alpha)$. Since Q is homogeneous of degree N , then p is of degree N , and we already showed that p has $N + 1$ distinct roots. Therefore p must be identically zero. In particular, $Q(x_1) = p(0) = 0$. Therefore, the polynomial Q vanishes on the dense subset S of $\mathbb{R}^n$ and this implies that Q is identically zero. ■

Definition 3.141. A subset of $\mathbb{R}^n$ is said to be an **algebraic variety** if it is the zero set of a family of polynomials. The algebraic variety is **proper** if it is not equal to $\mathbb{R}^n$.

The following result is a criterion for uniqueness of the k -plane transform which does not require data from all directions. It is by Smith, Solmon, and Wagner [566] in the case of the x-ray and Radon transform. Their result was extended by Keinert [348] to the k -plane transform. We present Keinert's result in the next theorem.

Theorem 3.142. Let $S \subset G_{k,n}$ be a set of k -dimensional subspaces of $\mathbb{R}^n$ such that the union of all $\eta^\perp$ with $\eta \in S$ is not contained in a proper algebraic subvariety of $\mathbb{R}^n$. Let f be any compactly supported L^1 function. Then if $P_\eta f = 0$ for every $\eta \in S$, it follows that $f = 0$.

Proof. Since $f \in L^1_0$ we can use lemma 3.139 to show that

$$\widehat{f}(\xi) = \sum_{j=0}^{\infty} a_j(\xi) \quad (3.100)$$

where each a_j is a homogeneous polynomial of degree j . Let V be the algebraic variety which is the intersection of the varieties $\{a_j = 0\}$. Then $\widehat{f}(\xi) = 0$ if and only if $\xi \in V$.

Now let $\eta \in S$. Then the hypothesis implies that $P_\eta f = 0$ and the slice-projection theorem implies that $\widehat{f}(\xi'') = 0$ for every $\xi'' \in \eta^\perp$. Hence $\eta^\perp \subset V$. Since this is true for every $\eta \in S$, then we see that $\bigcup_{\eta \in S} \eta^\perp \subset V$. So this union is contained in the algebraic variety V . By the hypothesis, this variety cannot be proper, and hence $V = \mathbb{R}^n$. But V is the set where $\widehat{f}$ is zero, so $\widehat{f}$, and hence f , vanishes on $\mathbb{R}^n$. ■

Corollary 3.143. Under the conditions of the preceding theorem, $f \in L^1_{loc}$ is uniquely determined by its projections $P_\eta f$ for $\eta \in S$.

Theorem 3.144 (Uniqueness theorem for the x-ray transform). If S is any infinite set of lines through the origin in $\mathbb{R}^n$, if $f \in L^1_0$, and if $X_\eta f$ vanishes for every $\eta \in S$, then $f = 0$.

Proof. As in the previous theorem, $\widehat{f}(\xi) = \sum_{j=0}^{\infty} a_j(\xi)$, where each a_j is a homogeneous polynomial of degree j and $\widehat{f}$ vanishes on infinitely many hyperplanes through

the origin (the hyperplanes $\eta^\perp$ orthogonal to the lines in S). Therefore, each homogeneous polynomial a_j vanishes on $j + 1$ such hyperplanes. By lemma 3.140 we see that each a_j and hence $\hat{f}$ is identically zero. The theorem follows immediately. ■

On the other hand, if the specified directions have the property that their orthogonal complements are contained in a proper algebraic subvariety, then there is an awesome amount of nonuniqueness. The following two results, which make this idea precise, were originally proved by Smith, Solmon, and Wagner [566] in the case of the x-ray transform. This generalization to the k -plane transform is from Keinert [348].

The first result is a special case of the second, but the second follows as a corollary. The intuitive meaning of the first result is that if we are given a finite set of directions and a sufficiently smooth, compactly supported object f , then we can find another smooth object with the same support which has vanishing x-rays from this finite set of directions yet which equals f on an arbitrarily large set. So the second object is like a “ghost” in the sense that x-rays just pass right through it without being absorbed at all.

Theorem 3.145. *Let $S \subset G_{k,n}$ be a set of k -dimensional subspaces of $\mathbb{R}^n$ with the property that there is a proper algebraic subvariety of $\mathbb{R}^n$ containing $\eta^\perp$ for every $\eta \in S$. Then if f is any function in C_0^∞ and K is any compact set in the interior of the support of f , then there is a function $h \in C_0^\infty$ such that*

$$\begin{aligned} P_\eta h &= 0, \text{ for every } \eta \in S, \\ h &= f \text{ on } K \text{ and} \\ \text{supp } h &\subset \text{supp } f. \end{aligned}$$

Proof. By the hypothesis and definition 3.141 there must be a nonzero polynomial Q , which vanishes on $\eta^\perp$ for every $\eta \in S$. We now use the celebrated theorem of Ehrenpreis [145] and Malgrange [418] on the existence of solutions to constant coefficient partial differential equations which states that if L is a constant coefficient differential operator and $g \in C_0^\infty(\mathbb{R}^n)$, then there is a C^∞ solution u to the partial differential equation $L(u) = g$. Helgason [291] has a conveniently accessible and short derivation of this theorem (compare, Hörmander [316] and Ortner and Wagner [460]).

According to the Ehrenpreis–Malgrange theorem, since $f \in C_0^\infty$, then there must be a C^∞ function w such that

$$Q(-i\partial)w = f.$$

Next, we can find a cutoff function $g \in C_0^\infty$ such that $g = 1$ on a neighborhood of K and $g = 0$ outside the support of f .

Define $h = Q(-i\partial)(gw)$. Then h agrees with f on a neighborhood of K since $gw = w$ there. Also $h \in C_0^\infty$ with support contained in the support of f . This proves two of the conclusions of the theorem and also guarantees that h has a Fourier transform. An easy consequence of the derivative theorem for the Fourier transform, specifically

formula (3.5), yields

$$\begin{aligned}\widehat{h}(\xi) &= (Q(-i\partial)(gw))^\wedge(\xi) \\ &= Q(\xi) \widehat{gw}(\xi).\end{aligned}$$

In particular, for $\eta \in S$ and $\xi'' \in \eta^\perp$, we can use the slice-projection theorem to get

$$\begin{aligned}\widehat{P_\eta h}(\xi'') &= (2\pi)^{\frac{k}{2}} \widehat{h}(\xi'') \\ &= (2\pi)^{\frac{k}{2}} Q(\xi'') \widehat{gw}(\xi'') \\ &= 0\end{aligned}$$

because Q vanishes on such $\eta^\perp$. Taking the inverse Fourier transform gives the remaining conclusion. ■

Theorem 3.146. *Let $S \subset G_{k,n}$ be a set of k -dimensional subspaces of $\mathbb{R}^n$ with the property that there is a proper algebraic subvariety of $\mathbb{R}^n$ containing $\eta^\perp$ for every $\eta \in S$. Then if f_0 is any function in C_0^∞ , f_1 is any function in C_0^∞ , and K is any compact set in the interior of the support of f_0 , there is a function $f \in C_0^\infty$ such that*

$$\begin{aligned}P_\eta f &= P_\eta f_0, \text{ for every } \eta \in S, \\ f &= f_1 \text{ on } K \text{ and} \\ \text{supp } f &\subset \text{supp } f_0.\end{aligned}$$

Proof. There is no loss of generality by using a cutoff function to ensure that f_1 is defined the same on K but has support in $\text{supp } f_0$. Then by using theorem 3.145 on both f_0 and f_1 we obtain functions h_0 and h_1 in C_0^∞ such that

$$\begin{aligned}P_\eta h_j &= 0, \text{ for every } \eta \in S, \\ h_j &= f_j \text{ on } K \text{ and} \\ \text{supp } h_j &\subset \text{supp } f_0.\end{aligned}$$

Now define $f = f_0 - h_0 + h_1$. Then since $P_\eta h_0 = P_\eta h_1 = 0$ we have

$$P_\eta f(x'') = P_\eta f_0(x'')$$

for $\eta \in S$, and $x'' \in \eta^\perp$. Also since $h_0 = f_0$ on K and $h_1 = f_1$ on K , then

$$f = f_0 - h_0 + h_1 = f_0 - f_0 + f_1 = f_1$$

on K . Finally, the fact that $\text{supp } f \subset \text{supp } f_0$ follows from the support conditions on f_0 , h_0 , and h_1 . ■

Corollary 3.147. *Let $\eta_1, \dots, \eta_m$ be a finite set of k -dimensional subspaces of $\mathbb{R}^n$. Then if f_0 is any function in C_0^∞ , f_1 is any function in C_0^∞ , and K is any compact set in the interior of the support of f_0 , there is a function $f \in C_0^\infty$ such that*

$$\begin{aligned}P_\eta f &= P_\eta f_0, \text{ for every } \eta \in S, \\ f &= f_1 \text{ on } K \text{ and} \\ \text{supp } f &\subset \text{supp } f_0\end{aligned}$$

Proof. An algebraic subvariety of $\mathbb{R}^n$ is the locus of a finite set of polynomial equations. Because each orthogonal complement $\eta_j^\perp$ is determined by a finite set of linear equations, then each $\eta_j^\perp$ is a proper algebraic subvariety of $\mathbb{R}^n$. The union of a finite number of proper algebraic subvarieties is also a proper algebraic subvariety, so the conditions of the theorem are satisfied and the corollary follows immediately. ■

The assertion of Smith, Solmon, and Wagner [566] that “A finite set of radiographs tells nothing at all” is a direct consequence of this corollary in the case $k = 1$, c.f. theorem 3.138, above.

The previous results apply to functions in rather large classes, say spaces containing all compactly supported C^∞ functions. Possibly, if we are more restrictive on the type of functions we are dealing with, a finite set of projections may determine a function. In fact Smith, Solmon, and Wagner [566] proved:

- For almost any finite-dimensional space of objects F , the objects in F can be distinguished by a single radiograph from almost any direction.

The precise statement of this result is:

Theorem 3.148 (Smith, Solmon, and Wagner [566]). *Let E be a finite dimensional subspace of L^1 , with dimension N , and let V be the set of directions such that at least two objects in E have the same x-ray projection from the direction θ . Then*

- (a) V is an algebraic variety in S^{n-1} .
- (b) $V = S^{n-1}$ if and only if there are polynomials $q_1, \dots, q_N$ such that

$$\sum_{j=1}^N q_j(\theta) P_\theta f_j(\xi'') = 0 \text{ for all } \theta \text{ and all } \xi'' \in \theta^\perp$$

Remark 3.149. Quoting from Smith, Solmon, and Wagner [566]: “Part (a) says that either no single direction serves to distinguish between the objects of F , or else almost any direction does that job; while part (b) says the the latter is the usual case.”

Indeed, if V is all of S^{n-1} , then for every direction $\theta \in S^{n-1}$ there are two distinct objects with the same x-ray projection in the direction θ . Hence, no single direction can distinguish objects. The only other alternative is that V is a proper subvariety in which case it has measure zero, so almost any direction will distinguish all objects.

The proof of theorem 3.148 is based on the following lemma. We refer the reader to Smith, Solmon, and Wagner [566] for the proof of the lemma.

Lemma 3.150. *Let $F_1, \dots, F_N$ be real analytic functions on $\mathbb{R}^n$ and let V be the set of directions θ such that the restrictions of the F_j to the hyperplane $\theta^\perp$ are linearly dependent. Then*

- (a) V is an algebraic variety in S^{n-1} .
- (b) $V = S^{n-1}$ if and only if there are polynomials $q_1, \dots, q_N$ such that

$$\sum_{j=1}^N q_j(\theta) F_j(\xi - \langle \xi, \theta \rangle \theta) = 0 \text{ for all } \theta \text{ and all } \xi$$

Once this lemma is proved, the proof of theorem 3.148 proceeds by choosing a basis of L^1 functions $f_1, \dots, f_N$ of E and defining $F_j = \widehat{f_j}$. Since f_j is L^1 , then F_j is real analytic. The lemma applies and we denote the algebraic variety V of this lemma by V_1 while we denote by V the set of directions such that at least two objects in E have the same x-ray projection from the direction θ .

Let f and $g \in E$. For a fixed direction θ , $P_\theta f = P_\theta g$ if and only if $P_\theta(f - g) = 0$ on $\theta^\perp$. By the slice-projection theorem, this happens if and only if $\widehat{(f - g)}(\xi'') = 0$ on $\theta^\perp$. Since $\widehat{(f - g)}$ is a linear combination of $F_1, \dots, F_N$, then $F_1, \dots, F_N$ are linearly dependent on the hyperplane $\theta^\perp$ so $\theta \in V_1$. Conversely, any element θ of V_1 has the property that at least two objects in E have the same x-ray projection from the direction θ , so $V = V_1$. We know from the lemma that V_1 is an algebraic variety in S^{n-1} , so part (a) of the theorem is proved. Part (b) follows from the observation that $\xi - \langle \xi, \theta \rangle \theta$ is the orthogonal projection of ξ onto $\theta^\perp$. Therefore, by the slice-projection theorem and the relation $F_j = \widehat{f_j}$ we get

$$\sum_{j=1}^N q_j(\theta) F_j(\xi - \langle \xi, \theta \rangle \theta) = (2\pi)^{-\frac{k}{2}} \sum_{j=1}^N q_j(\theta) \widehat{P_\theta f_j}(\xi - \langle \xi, \theta \rangle \theta)$$

Since any $\xi'' \in \theta^\perp$ is of the form $\xi - \langle \xi, \theta \rangle \theta$, and vice versa, we see in that condition that polynomials $q_1, \dots, q_N$ exist such that

$$\sum_{j=1}^N q_j(\theta) P_\theta f_j(\xi'') = 0 \text{ for all } \theta \text{ and all } \xi'' \in \theta^\perp$$

is equivalent to the condition that there exist polynomials $q_1, \dots, q_N$ such that

$$\sum_{j=1}^N q_j(\theta) F_j(\xi - \langle \xi, \theta \rangle \theta) = 0 \text{ for all } \theta \text{ and all } \xi$$

By the lemma this is equivalent to $V_1 = S^{n-1}$. Since we already proved that $V = V_1$, the demonstration is complete. ■

3.13 Additional References and Results

3.13.1 General

Fuglede [183] gave perhaps the earliest treatment of the k -plane transform, although he did not refer to it by this (or indeed any) name. He also gave an inversion formula similar to ours but without a rigorous discussion of the types of functions for which it is valid. In this paper is also a very nice and succinct treatment of Grassmann manifolds as homogeneous spaces. Fuglede's formula, in our notation, where P is the k -plane transform, is

$$\int_{A^k} P f(\eta, 0) d\eta = \frac{|S^{k-1}|}{|S^{n-1}|} \int_{\mathbb{R}^n} |w|^{k-n} f(w) dw$$

where A^k is the Grassmannian $G_{k,n}$ with the Haar measure chosen so that $|A^k| = 1$. By translating f and taking into account the different normalization of our Haar measure we can replace the origin by an arbitrary $x \in \mathbb{R}^n$ and thereby get Fuglede's theorem as stated previously.

Gindikin, Kirillov, and Fuks [210] give a survey of Gel'fand's work on the Radon transform. Gel'fand, Gindikin, and Graev [187] give a basic introduction to both the Radon transform in $\mathbb{R}^n$ and to the k -plane transform.

Calderón [74], which we already mentioned in connection with extending the domain of the k -plane transform, is a survey paper with a different method for defining the k -plane transform.

The book [377] by Lavrent'ev and Savel'ev is a good reference for some of the functional analytic tools used in tomography. It also contains an interesting appendix on inversion formulas in tomography that was written by A. L. Bukhgeim. The reference [377] is an English translation of [376].

3.13.2 Remarks on the Riesz Multiplier Theorem and the Inversion Theorem

The original formulation of the Riesz multiplier theorem 3.74 was by Smith and Solmon [565] who proved it under the conditions that g be an L^2 function with the additional property that $(1 + |x|)^{k-n}g(x)$ is integrable and that either side of the equation $\widehat{R_k * f}(\xi) = |\xi|^{-k} \widehat{f}(\xi)$ be square integrable.

Smith and Keinert [564] proved both the Riesz multiplier theorem for $k = 1$ and the inversion theorem, theorem 3.79, for the x-ray transform. Keinert [348] generalized this result to the k -plane transform. Their proofs were fairly complicated, involving the Calderón–Zygmund theory of singular integrals [75] and some other deep theorems from real analysis. Our proof is much more elementary.

As mentioned in section 3.9, Solmon [569] developed an inversion formula for the k -plane transform based on extending a densely defined unbounded operator on $L^2(\mathbb{R}^n)$. His inversion formula is equivalent to equation (3.88) in theorem 3.106, but the constants are different because Solmon normalizes the Grassmannian Haar measure differently. This paper has a wealth of other results on the k -plane transform.

Takiguchi and Kaneko [595] extend the basic theory of the Radon transform to the case of hyperfunctions.

3.13.3 Uniqueness and Nonuniqueness

We know that if the Radon transform of an integrable function is identically zero, then the function must be identically zero (theorem 3.131). Falconer [156] proved the following rather curious result: if n is odd, and the Radon transform of an integrable function supported on a compact convex set is constant, then the function must be identically zero. He also shows that this result is false in even dimensions.

Concerning theorem 3.148 on the possibility of determining a function from a single projection, the paper [359] by Kuba and Volčič and the paper [358] by Kuba on determining characteristic functions of measurable subsets of $\mathbb{R}^2$ from two projections may be of interest.

Solmon [572] showed that two functions with the same divergent beam x-ray transform from a finite set of sources may differ arbitrarily on any compact set in the interior of their support (compare, section 5.10.2).

3.13.4 Mixed Norm Estimates

Wang [617] has a mixed norm estimate for the x-ray transform restricted to certain Kirillov-type curves. We refer the reader to [617] for the details. Results on mixed-norm estimates for a certain restricted x-ray transform may be found in Wolff [623], Erdoğan [151], and Christ and Erdoğan [92].

Seeger [548] and Lee [384]⁸, discuss mixed-norm estimates for the generalized Radon transform introduced by Phong and Stein [491]. Compare, Bak, Oberlin, and Seeger [32].

Drury [133] showed that the k -plane transform P is a bounded operator between the Lorentz space $L(\frac{n}{k}, 1)$ and the mixed-norm space $L^n(L^\infty)$ provided $n \geq 3$ and $k > \frac{n}{2}$. Drury [134] generalized Christ's paper [91] to the case of the Radon transform between Grassmannians (compare, section 5.10.4).

It is a consequence of Christ's theorem that for each $q \in [1, n+1]$, there exists a unique $p \in [1, \frac{n+1}{k+1}]$ such that the k -plane transform is a bounded linear operator $L^p(\mathbb{R}^n) \rightarrow L^q(\mathcal{G}_{k,n})$. In fact, $p = \frac{nq}{n-k+kq}$. The question arises as to whether one can find the exact value of the operator norm $\|P\|$ in this case. Baernstein and Loss [29] conjectured that $\|P\| = \frac{\|P f_0\|_{L^q(\mathcal{G}_{k,n})}}{\|f_0\|_{L^p(\mathbb{R}^n)}}$, where $f_0(x) = (a + b|x|^2)^{-\frac{1}{2} \frac{n-k}{p-1}}$, with a, b any positive constants. They prove the truth of their conjecture in the case in which either $q = 2$ or $k = 2$ and q is an integer in $[1, n+1]$. They also have an analogous conjecture and partial results for the Riemann–Liouville fractional integral operators.

Greenleaf and Uhlmann [234] have some estimates of singular Radon transforms.

Sogge and Stein [567] give maximal function and L^p estimates for a generalized Radon transform which integrates over hypersurfaces in $\mathbb{R}^n$. These estimates hold for large enough p in dimension $n \geq 6$ provided the Gaussian curvature of the surfaces does not vanish to infinite order. In [568] the authors extend these results to $n \geq 3$, where the curvature condition is reformulated in terms of rotational curvature. This type of curvature is equivalent to a smoothness and nondegeneracy condition formulated in Greenleaf and Uhlmann [234].

Cnops [99] gives conditions under which the Radon projection R_θ is a continuous operator between a weighted L^p -space and a weighted L^q -space.

3.13.5 Geometric Tomography

Let X be a subset of $\mathbb{R}^n$. The x-ray projection of the characteristic function of X is then the same as the length of the intersection of a line in the direction of the x-ray with X . Geometric tomography is concerned with reconstructing subsets of $\mathbb{R}^n$ from

⁸ The paper [384] by Lee originally appeared as Lee [385], but unfortunately the references were not defined and appeared instead as question marks. Therefore the publisher reprinted the full paper, including correct references, as Lee [384]. See the publisher's erratum [386].

these lengths. In addition to the results of this section, the reader may consult Gardner's book [185] for a detailed treatment of these ideas.

Lorentz [395] gave conditions for two functions $P(x)$ and $Q(y)$ to be the projections, parallel to the coordinate axes, of a subset X of the plane. Letting p and q represent the nonincreasing rearrangements of P and Q , respectively, the conditions are

$$\begin{aligned} \int_0^x p(u) du &\leq \int_0^x q^{-1}(u) du \quad \text{for all } x > 0 \text{ and} \\ \int_0^y q(u) du &\leq \int_0^y p^{-1}(u) du \quad \text{for all } y > 0. \end{aligned}$$

Furthermore, Lorentz proved that the two projections uniquely determine a subset X of the plane if and only if $q = p^{-1}$. Kuba and Volčič [359] reformulated Lorentz's conditions. In the case that the projections uniquely determine a subset, Kuba and Volčič provide a reconstruction algorithm for obtaining the set from the two projections.

Fishburn et al. [175] consider Lorentz's uniqueness problem in $\mathbb{R}^n$. They define an additive set to be the locus of points in $\mathbb{R}^n$ satisfying $f_1(x_1) + f_2(x_2) + \cdots + f_n(x_n) \geq 0$, where each f_i is bounded and measurable on $\mathbb{R}$. They prove that for an additive set, the Radon projections parallel to the coordinate axes uniquely determine the set. In the case of $\mathbb{R}^2$, Fishburn et al. show that a set is uniquely determined by projections parallel to the coordinate axes if and only if it is an additive set.

It is known that two, and even three, directions in a plane do not uniquely determine a set in $\mathbb{R}^n$, even a convex set. Hammer's problem [256] is: "How many X-ray pictures of a convex body must be taken to permit its exact reconstruction?" Gardner and McMullen address Hammer's problem. In [186] they are able to prove that, up to translation, convex bodies in $\mathbb{R}^n$ may be distinguished by four x-ray projections, with transcendental cross ratios, provided the directions lie in a fixed plane in $\mathbb{R}^n$. This result, which can be considered to be a uniqueness result for a type of finite x-ray transform, is a consequence of the main theorem in [186], which states the same result for any set of directions in a plane contained in $\mathbb{R}^n$ that is not linearly equivalent to a subset of the directions of the set of diagonals of a regular polygon. Also see Kincses and Kurusa [352] and Kurusa [374] for some generalizations.

3.14 Appendix

In this appendix we provide the demonstrations of some lemmas that have been postponed from the statements in the main body of the chapter. Constants in the body of a proof may take different values at different stages of the proof. In other words we let the same symbol c stand for different values in an expression, as long as these values are all constants.

3.14.1 Proof of Lemma 3.34

Lemma 3.34. *Let $\beta > n - k$. If $\psi(r) = \int_0^1 (1 + rt)^{-\beta} (1 - t^2)^{\frac{k}{2}-1} t^{n-k-1} dt$, then there exists a constant c independent of β, k, n such that $|\psi(r)| \leq c(1 + r)^{k-n}$.*

Proof. Define $\psi(r; a, b) = \int_a^b (1+rt)^{-\beta} (1-t^2)^{\frac{k}{2}-1} t^{n-k-1} dt$.

For now assume that $r > 2$. We show that each of the integrals $\psi(r; 0, \frac{1}{r})$, $\psi(r; \frac{1}{r}, \frac{1}{2})$, and $\psi(r; \frac{1}{2}, 1)$ is bounded by a constant multiple of r^{k-n} .

Note that $(1-t^2)^{\frac{k}{2}-1}$ is continuous on the interval $[0, \frac{1}{2}]$ so there is a constant c_1 depending only on k , such that $|(1-t^2)^{\frac{k}{2}-1}| = (1-t^2)^{\frac{k}{2}-1} \leq c_1$ on this interval. Also note that $(1+rt)^{-\beta} \leq 1$ on $[0, 1]$.

Therefore we get

$$\begin{aligned} \psi\left(r; 0, \frac{1}{r}\right) &= \int_0^{\frac{1}{r}} (1+rt)^{-\beta} (1-t^2)^{\frac{k}{2}-1} t^{n-k-1} dt \\ &\leq c_1 \int_0^{\frac{1}{r}} t^{n-k-1} dt = c_1 r^{k-n} \end{aligned}$$

In the next two integrals we have $t > 0$ and since $\beta > n-k > 0$ we see that $(1+rt)^{-\beta} \leq (rt)^{-\beta}$ and we can factor $r^{-\beta}$ out of the integral, leaving an integrand of $t^{n-k-\beta-1} (1-t^2)^{\frac{k}{2}-1}$.

In the integral defining $\psi(r; \frac{1}{r}, \frac{1}{2})$ we also have $(1-t^2)^{\frac{k}{2}-1} \leq c_1$ so we get $\psi(r; \frac{1}{r}, \frac{1}{2}) \leq c_1 r^{-\beta} \int_{\frac{1}{r}}^{\frac{1}{2}} t^{n-k-\beta-1} dt$ which we can easily show is bounded by $c_2 r^{k-n} (1 - (\frac{r}{2})^{n-k-\beta})$. But since $n-k-\beta < 0$ and $r > 2$, then $(1 - (\frac{r}{2})^{n-k-\beta}) \leq 1$ so we get

$$\psi\left(r; \frac{1}{r}, \frac{1}{2}\right) \leq c_2 r^{-\beta} \leq c_2 r^{k-n}$$

since $\beta > n-k$.

Finally, $\psi(r; \frac{1}{2}, 1) \leq c_1 r^{-\beta} \int_{\frac{1}{2}}^1 t^{n-k-\beta-1} (1-t^2)^{\frac{k}{2}-1} dt$. Since $\beta > n-k$, then $r^{-\beta} \leq r^{k-n}$. Also the preceding integral converges for our choice of β relative to n and k and is a constant c_2 with respect to all three of β, n, k . Therefore, $\psi(r; \frac{1}{2}, 1) \leq c_3 r^{k-n}$.

Putting together all these integrals shows that $|\psi(r)| \leq c(1+r)^{k-n}$, if $r > 2$. It is easy to check that $\psi(r)/(1+r)^{k-n}$ is continuous on $[0, \infty)$. Hence for r on the set $[0, 2]$ the estimate $\psi(r) \leq c(1+r)^{k-n}$ is completely trivial. Also it is obvious that the estimate $\psi(r) \leq c r^{k-n}$ is equivalent to the estimate $\psi(r) \leq c(1+r)^{k-n}$ if $r > 2$. This all means that there is a suitable constant satisfying the specified conditions such that $|\psi(r)| \leq c(1+r)^{k-n}$. ■

3.14.2 Proof of Lemma 3.60

Lemma 3.60. Let $0 < \alpha < n$. There is a constant C such that for any measurable function ϕ and any $t \in \mathbb{R}^n$ we have

$$|R_\alpha * \phi(t)| \leq C(1+|t|)^{\alpha-n} (\|\phi\|_{L^1} + \|(1+|x|)^n \phi(x)\|_{L^\infty})$$

The constant depends only on n, α .

Proof. It suffices to prove this estimate for the function $J(t) = J = \int_{\mathbb{R}^n} |t - y|^{\alpha-n} |\phi(y)| dy$. We consider two cases, $|t| > 1$ and $|t| \leq 1$. In each case we split the integral J into pieces J_l . It suffices to prove an estimate either of the form $J_l \leq C(1 + |t|)^{\alpha-n} \|\phi\|_{L^1}$ or of the form $C(1 + |t|)^{\alpha-n} \|(1 + |x|)^n \phi(x)\|_{L^\infty}$ for each subintegral.

Case 1: $|t| > 1$.

Partition $\mathbb{R}^n$ into the following three disjoint regions:

$$\begin{aligned} A_1 : |y| &\leq \frac{|t|}{2} \\ A_2 : \frac{|t|}{2} &< |y| < 2|t| \\ A_3 : |y| &\geq 2|t|. \end{aligned}$$

and split the integral J into three corresponding integrals $J_1, \dots, J_3$.

On A_1 we have $|t - y| \geq |t| - |y| \geq \frac{|t|}{2}$ so $|t - y|^{\alpha-n} \leq (\frac{|t|}{2})^{\alpha-n}$ because $\alpha - n < 0$. Hence, there is a constant C depending only on α and n such that

$$\begin{aligned} J_1 &= \int_{|y| \leq \frac{|t|}{2}} |t - y|^{\alpha-n} |\phi(y)| dy \leq \left(\frac{|t|}{2}\right)^{\alpha-n} \int_{|y| \leq \frac{|t|}{2}} |\phi(y)| dy \\ &\leq C |t|^{\alpha-n} \|\phi\|_{L^1} \end{aligned} \quad (3.101)$$

On A_3 we have $|t - y| \geq |y| - |t| \geq |t|$ so in a similar fashion we get

$$J_3 \leq C |t|^{\alpha-n} \|\phi\|_{L^1} \quad (3.102)$$

On the remaining region, A_2 :

$$J_2 = \int_{\frac{|t|}{2} < |y| < 2|t|} |t - y|^{\alpha-n} |\phi(y)| dy.$$

There is a $\theta \in S^{n-1}$ such that $t = |t|\theta$. Make the substitution $y = |t|(\theta - v)$ in J_2 . This yields

$$J_2 = |t|^\alpha \int_{\frac{1}{2} < |\theta - v| < 2} |v|^{\alpha-n} |\phi(|t|(\theta - v))| dv \quad (3.103)$$

Now

$$\begin{aligned} |\phi(|t|(\theta - v))| &= (|t| |\theta - v|)^{-n} (|t| |\theta - v|)^n |\phi(|t|(\theta - v))| \\ &\leq |t|^{-n} |\theta - v|^{-n} (1 + (|t| |\theta - v|))^n |\phi(|t|(\theta - v))| \end{aligned}$$

For almost any $u \in \mathbb{R}^n$ we have $|1 + |u||^n |\phi(u)| \leq \|(1 + |x|)^n \phi(x)\|_{L^\infty}$. Applying this to $u = |t|(\theta - v)$ and using the last inequality gives

$$\begin{aligned} |\phi(|t|(\theta - v))| &\leq |t|^{-n} |\theta - v|^{-n} \|(1 + |x|)^n \phi(x)\|_{L^\infty} \\ &\leq |t|^{-n} 2^n \|(1 + |x|)^n \phi(x)\|_{L^\infty} \end{aligned}$$

since the condition $|\theta - v| > \frac{1}{2}$ implies $|\theta - v|^{-n} < 2^n$. Using this estimate in equation (3.103), together with the observation that the set where $\frac{1}{2} < |\theta - v| < 2$ is contained in the set where $|v| < 3$, we get $J_2 \leq |t|^{\alpha-n} 2^n \|(1 + |x|)^n \phi(x)\|_{L^\infty} \int_{|v|<3} |v|^{\alpha-n} dv$, so there is a constant $C = 2^n \int_{|v|<3} |v|^{\alpha-n} dv$, which depends only on α and n , such that

$$J_2 \leq C |t|^{\alpha-n} \|(1 + |x|)^n \phi(x)\|_{L^\infty} \quad (3.104)$$

Adding the estimates for the three integrals, (3.101), (3.102), and (3.104) shows that there is a constant C depending only on α and n such that

$$J \leq C |t|^{\alpha-n} (\|\phi\|_{L^1} + \|(1 + |t|)^n \phi(x)\|_{L^\infty})$$

But $|t| > 1$ so we can modify the constant to get

$$J \leq C (1 + |t|)^{\alpha-n} (\|\phi\|_{L^1} + \|(1 + |t|)^n \phi(x)\|_{L^\infty})$$

Case 2: $|t| \leq 1$.

In this case $(1 + |t|)^{\alpha-n}$ is a nonvanishing continuous function on a compact set and therefore has a minimum $m > 0$. Define $c_1 = \frac{1}{m}$ noting then that $1 = c_1 m \leq c_1 (1 + |t|)^{\alpha-n}$ and c_1 is a constant depending only on α and n .

Decompose J as follows:

$$\begin{aligned} J &= \int_{|t-y|\leq 1} |t-y|^{\alpha-n} |\phi(y)| dy + \int_{|t-y|>1} |t-y|^{\alpha-n} |\phi(y)| dy \\ &= J_1 + J_2 \end{aligned}$$

By definition of the L^∞ norm we see that

$$J_1 \leq \|\phi\|_{L^\infty} \int_{|t-y|\leq 1} |t-y|^{\alpha-n} dy$$

But $\|\phi\|_{L^\infty} \leq \|(1 + |x|)^n \phi(x)\|_{L^\infty}$, and the substitution $w = t - y$ together with $\alpha - n > -n$ shows that $\int_{|t-y|\leq 1} |t-y|^{\alpha-n} dy$ is a constant c_2 , depending only on α and n . Therefore,

$$\begin{aligned} J_1 &\leq \|(1 + |x|)^n \phi(x)\|_{L^\infty} c_2 (c_1 m) \\ &\leq c_2 c_1 \|(1 + |x|)^n \phi(x)\|_{L^\infty} (1 + |t|)^{\alpha-n}. \end{aligned}$$

Also $|t - y|^{\alpha-n} < 1$ if $|t - y| > 1$ so

$$\begin{aligned} J_2 &\leq c_1 m \int_{|t-y|>1} |\phi(y)| dy \\ &\leq c_1 (1 + |t|)^{\alpha-n} \|\phi\|_{L^1}. \end{aligned}$$

■

3.14.3 Proof of Lemma 3.71

Lemma 3.71. *If $\varphi \in \mathcal{S}$ and $\partial^\beta \varphi(0) = 0$ for all multi-indices with $|\beta| < k$, then there are functions φ_α , also in the Schwartz class $\mathcal{S}$, such that for any $\xi \in \mathbb{R}^n$ we have*

$$\varphi(\xi) = \sum_{|\alpha|=k} \xi^\alpha \varphi_\alpha(\xi)$$

Proof. Let us begin with the following elementary observations.

1. $\varphi(0) = 0$ because of the hypothesis and the fact that $\partial^\alpha \varphi(0) = \varphi(0)$ if α is the zero multi-index.
2. $\lim_{|b| \rightarrow \infty} \varphi(b) = 0$ because of the definition of the Schwartz class of rapidly decreasing functions.
3. If $\varphi \in \mathcal{S}$ and $\varphi(0) = 0$, then $\varphi(\xi) = - \int_1^\infty \sum_{j=1}^n \xi_j \frac{\partial \varphi}{\partial \xi_j}(t_1 \xi) dt_1$

Observation (3) is an easy consequence of the following chain-rule calculation

$$\frac{\partial}{\partial t_1} (\varphi(t_1 \xi)) = \sum_{j=1}^n \xi_j \frac{\partial \varphi}{\partial \xi_j}(t_1 \xi)$$

from which it follows that

$$\begin{aligned} \int_1^\infty \sum_{j=1}^n \xi_j \frac{\partial \varphi}{\partial \xi_j}(t_1 \xi) dt_1 &= \int_1^\infty \frac{\partial}{\partial t_1} (\varphi(t_1 \xi)) dt_1 \\ &= \lim_{b \rightarrow \infty} \varphi(b \xi) - \varphi(\xi) \end{aligned}$$

If $\xi = 0$, then by observation (1) we see that $\lim_{b \rightarrow \infty} \varphi(b \xi) - \varphi(\xi) = 0 = -\varphi(\xi)$, whereas if $\xi \neq 0$, then $|b \xi| \rightarrow \infty$ as $b \rightarrow \infty$, so by observation (2) we again arrive at $-\varphi(\xi)$.

Now define the differential operator $L = (\xi_1 \frac{\partial}{\partial \xi_1} + \cdots + \xi_n \frac{\partial}{\partial \xi_n})$. We will prove the following result by mathematical induction:

- If $\varphi \in \mathcal{S}$ and $\partial^\alpha \varphi(0) = 0$ for all multi-indices with $|\alpha| < k$, then

$$\int_1^\infty \cdots \int_1^\infty L^k(\varphi)(t_1 t_2 \cdots t_k \xi) dt_1 dt_2 \cdots dt_k = (-1)^k \varphi(\xi) \quad (3.105)$$

The initial step of the induction is a consequence of observation (3).

Assuming that equation (3.105) is true for all orders less than k , define

$$g(\xi) = \int_1^\infty \cdots \int_1^\infty L^{k-1}(\varphi)(t_1 t_2 \cdots t_{k-1} \xi) dt_1 dt_2 \cdots dt_{k-1}$$

An immediate consequence of the induction hypothesis is that $g(\xi) = (-1)^{k-1} \varphi(\xi)$. Hence, $g \in \mathcal{S}$ and since $g(0) = 0$, we can apply the induction hypothesis with $k = 1$

to obtain

$$\int_1^\infty Lg(t_k \xi) dt_k = -g(\xi) = (-1)^k \varphi(\xi) \quad (3.106)$$

Note that the integral operators $\int_1^\infty dt_j$ commute with the differential operator L , so by equation (3.106) and the definition of g we get

$$\begin{aligned} (-1)^k \varphi(\xi) &= \int_1^\infty Lg(t_k \xi) dt_k \\ &= \int_1^\infty \cdots \int_1^\infty L^k(\varphi)(t_1 t_2 \cdots t_k \xi) dt_1 dt_2 \cdots dt_k \end{aligned}$$

which proves (3.105).

An application of the multinomial formula demonstrates that

$$\begin{aligned} L^k &= \left(\xi_1 \frac{\partial}{\partial \xi_1} + \cdots + \xi_n \frac{\partial}{\partial \xi_n} \right)^k \\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha!} \xi^\alpha \partial^\alpha \end{aligned}$$

and applying this to (3.105) shows that if we define

$$\varphi_\alpha(\xi) = (-1)^k \frac{k!}{\alpha!} \int_1^\infty \cdots \int_1^\infty \frac{\partial^\alpha \varphi}{\partial \xi^\alpha}(t_1 t_2 \cdots t_k \xi) dt_1 dt_2 \cdots dt_k \quad (3.107)$$

then $\varphi(\xi) = \sum_{|\alpha|=k} \xi^\alpha \varphi_\alpha(\xi)$.

It remains to show that each $\varphi_\alpha \in \mathcal{S}$. Because $\varphi \in \mathcal{S}$, we can take the derivative out of the integrals in equation (3.107) and it suffices to show that a function of the form

$$\int_1^\infty \cdots \int_1^\infty \varphi(t_1 t_2 \cdots t_k \xi) dt_1 dt_2 \cdots dt_k$$

is in the Schwartz class provided that $\varphi \in \mathcal{S}$ and $\varphi(0) = 0$. In the sequel we always make the assumption that all functions f in the Schwartz class that are being integrated satisfy the condition $f(0) = 0$.

Remark 3.151. Note that if $\varphi \in \mathcal{S}$ and $\varphi(0) \neq 0$, then this integral is not in the Schwartz class since it will have an infinite value at $\xi = 0$.

Let us prove the following result.

- If $f \in \mathcal{S}$, and $f(0) = 0$, then the function $g(x) = \int_1^\infty f(tx) dt$ is also in $\mathcal{S}$ and $g(0) = 0$.

Observe that derivatives can be taken inside the integral. This is because the assumption $f \in \mathcal{S}$ implies that for any multi-indices α and any nonnegative integer m there is a constant $C_{\alpha,m}$ such that

$$|w|^m |\partial^\alpha f(w)| \leq C_{\alpha,m}$$

Now $\partial^\alpha(f(tx)) = t^{|\alpha|} (\partial^\alpha f)(tx)$ so from the preceding inequality with $m = |\alpha| + 2$ and for $t \geq 1$ we get

$$\begin{aligned} |\partial^\alpha(f(tx))| &\leq t^{|\alpha|} C_{\alpha,m} \frac{1}{|tx|^m} \\ &\leq \frac{C_{\alpha,m}}{|x|^m} \frac{1}{t^2} \end{aligned} \quad (3.108)$$

This shows that the integrand in $\int_1^\infty \partial^\alpha(f(tx)) dt$ is integrable and hence that $\partial^\alpha g(x) = \int_1^\infty \partial^\alpha(f(tx)) dt$.

Using a similar argument, for any multi-indices α, β we have

$$|x|^{j+2} |(\partial^\alpha f)(tx)| \leq C_{\alpha,j+2} \frac{1}{t^{j+2}}$$

If $|x| \geq 1$ this implies

$$|x|^j |(\partial^\alpha f)(tx)| \leq C_{\alpha,j+2} \frac{1}{t^{j+2}}$$

and upon integrating

$$|x|^j \left| \int_1^\infty (\partial^\alpha f)(tx) dt \right| \leq \text{const}$$

and this proves the required estimate, $|x|^j |\partial^\alpha g(x)| \leq \text{const}$, if $|x| \geq 1$. If $0 < |x| < 1$, then

$$|x|^j \left| \int_1^\infty (\partial^\alpha f)(tx) dt \right| \leq \left| \int_1^\infty (\partial^\alpha f)(tx) dt \right| \leq \text{const}$$

by (3.108). We have thus shown that $g \in \mathcal{S}$. Also $g(0) = \int_1^\infty f(0) dt = 0$ by the assumption that $f(0) = 0$.

It now follows that $\varphi_\alpha \in \mathcal{S}$, since φ_α is an iteration of functions of the type $\int_1^\infty \partial^\alpha f(tx) dt$. ■

3.14.4 Proof of Lemma 3.72

Lemma 3.72. *Let $0 < k < n$ and let $e \in L_0^\infty$. Define $e_r(x) = r^{-n} e(\frac{x}{r})$ for $r > 0$. Then*

1. If $\widehat{u} \in L^1 + L^2$, then $\langle u, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$.
2. If $(1 + |x|)^{k-n} f(x) \in L^1$, then $\langle R_k * f, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$.

Proof. Observe that $e \in L^1$ since e is bounded with compact support. An easy consequence of the change of variables formula for integrals is that

$$\begin{aligned}\|e_r\|_{L^1} &= \|e\|_{L^1} \text{ and} \\ \|e_r\|_{L^2} &\rightarrow 0 \text{ as } r \rightarrow \infty\end{aligned}$$

Also by the dilation theorem for Fourier transforms we have $\widehat{e_r}(y) = \widehat{e}(ry)$.

It is surely enough to prove part one separately for $\widehat{u} \in L^1$ and then for $\widehat{u} \in L^2$.

If $\widehat{u}$ is L^1 , we set $f = \widehat{u} \in L^1$ and $g = e_r$, which is also in L^1 , and then we use the multiplication theorem for the Fourier transform (Stein and Weiss [583], page 8) which states: $\int_{\mathbb{R}^n} \widehat{f}(x) g(x) dx = \int_{\mathbb{R}^n} f(x) \widehat{g}(x) dx$. In our case this translates to $\langle u, e_r \rangle = \int_{\mathbb{R}^n} \widehat{e}(rx) \widehat{u}(x) dx$. Since e is integrable, we can apply the Riemann–Lebesgue lemma to get $\lim_{r \rightarrow \infty} \widehat{e}(rx) = 0$ for every fixed $x \neq 0$. But the boundedness of $\widehat{e}$ combined with the fact that $\widehat{u}$ is L^1 allows the use of the Lebesgue-dominated convergence theorem, so the last integral also approaches 0. This means that $\langle u, e_r \rangle \rightarrow 0$ as $r \rightarrow \infty$ if $\widehat{u} \in L^1$.

Now assume that $\widehat{u} \in L^2$. Then u is also in L^2 as are the functions e_r so by the Cauchy-Schwarz inequality we have

$$|\langle u, e_r \rangle| \leq \|u\|_{L^2} \|e_r\|_{L^2} \rightarrow 0 \text{ as } r \rightarrow \infty$$

as observed at the beginning of the proof.

Now we prove the second statement of the lemma. Let us denote the Lebesgue space $L^1((1 + |x|)^{k-n} dx)$ by the symbol $\mathcal{D}$. Then $\mathcal{D}$ is a Banach space under the norm $\|f\|_{\mathcal{D}} = \|(1 + |x|)^{k-n} f(x)\|_{L^1}$ and the subspace of functions with compact support is dense in $\mathcal{D}$.

There is no loss in generality by assuming that e is supported on the unit ball. Also

$$\begin{aligned}|\langle R_k * f, e_r \rangle| &\leq \int_{\mathbb{R}^n} (R_k * |f|)(x) r^{-n} e\left(\frac{x}{r}\right) dx \\ &\leq \|e\|_{L^\infty} \cdot \int_{|x| \leq r} (R_k * |f|)(x) r^{-n} dx \\ &= c \cdot |\langle R_k * |f|, \chi_r \rangle| \end{aligned}$$

where χ is the characteristic function of the unit ball. Therefore, if we could prove part 2 of the lemma with $e = \chi$, then it would be true for the original e also. Therefore, there is no loss in generality by assuming that e is the characteristic function of the unit ball and we make that assumption from here on.

The strategy is to consider the linear functionals $f \mapsto \langle R_k * f, e_r \rangle$ defined on the Banach space $\mathcal{D}$ for $r > 1$. We will show that these functionals are uniformly bounded in r on $\mathcal{D}$. Then we prove that for every g in a dense subspace of $\mathcal{D}$, the limits of these functionals as $r \rightarrow \infty$ is zero. This is enough to prove the result.

By theorem 3.62 we have

$$|\langle R_k * f, e_r \rangle| \leq C \left\| (1 + |x|)^{k-n} f(x) \right\|_{L^1} \left[\|e_r\|_{L^1} + \|(1 + |x|)^n e_r(x)\|_{L^\infty} \right].$$

Let r be an arbitrary real number greater than 1. Since $\|e_r\|_{L^1} = \|e\|_{L^1}$, which is constant with respect to r and since it is easy to show that $\|(1 + |x|)^n e_r(x)\|_{L^\infty}$ is uniformly bounded with respect to r by a constant, then we see that there is a constant K such that

$$|\langle R_k * f, e_r \rangle| \leq K \|f\|_{\mathcal{D}}.$$

This shows that the desired functionals are well defined and uniformly bounded in r , for sufficiently large r .

Let $\mathcal{C}$ denote the dense subspace of $\mathcal{D}$ consisting of functions with compact support. For the moment fix such a function f . For this f there is a constant b such that $\text{supp}(f) \subset B(b)$. There is no loss in generality by assuming $b > 1$. We will now prove that $\lim_{r \rightarrow \infty} |\langle R_k * f, e_r \rangle| = 0$.

First, the definition of the Riesz potential implies that $|R_k * f(x)| \leq c \cdot \int_{|y| \leq b} \frac{|f(y)|}{|x-y|^{n-k}} dy$. If $|x| \geq 2b$ then $|x-y| \geq |x| - |y| \geq |x| - b \geq \frac{|x|}{2}$, so $|R_k * f(x)| \leq c \cdot |x|^{k-n} \int_{|y| \leq b} |f(y)| dy$. Since f is fixed with respect to the limit, we can include the last integral in the constant thus arriving at

$$|R_k * f(x)| \leq c \cdot |x|^{k-n} \quad (3.109)$$

if $|x| \geq 2b$.

Now break up the integral defining the inner product $\langle R_k * f, e_r \rangle$ into an integral over the set $|x| \leq 2b$ plus an integral over its complement. An easy calculation involving the definition of e_r , where e is the characteristic function of the unit ball shows that if $r > 2b$, then

$$|\langle R_k * f, e_r \rangle| \leq r^{-n} \int_{|x| \leq 2b} |R_k * f(x)| dx + r^{-n} \int_{2b < |x| \leq r} |R_k * f(x)| dx \quad (3.110)$$

Since $(1 + |x|)^{k-n} f(x) \in L^1$, then we can use theorem 3.64 to show that $R_k * f$ is locally integrable. Hence, the first integral in inequality (3.110) is a finite constant independent of r . The second integral can be estimated using inequality (3.109): $\int_{2b < |x| \leq r} |R_k * f(x)| dx \leq c \cdot \int_{2b < |x| \leq r} |x|^{k-n} dx$. Changing to polar coordinates shows that $\int_{2b < |x| \leq r} |x|^{k-n} dx \leq c \cdot r^k$, so we also have $\int_{2b < |x| \leq r} |R_k * f(x)| dx \leq c \cdot r^k$. Using these results in (3.110) gives

$$|\langle R_k * f, e_r \rangle| \leq c \cdot (r^{-n} + r^{k-n}) \rightarrow 0 \text{ as } r \rightarrow \infty.$$

since $k < n$.

So now we know that the functionals $\langle R_k * f, e_r \rangle$ are uniformly bounded, by the constant K , for any $f \in \mathcal{D}$ as $r \rightarrow \infty$ and that for any fixed compactly supported function $f \in \mathcal{D}$, $\lim_{r \rightarrow \infty} \langle R_k * f, e_r \rangle = 0$. We now proceed to prove that for any $f \in \mathcal{D}$, $\lim_{r \rightarrow \infty} \langle R_k * f, e_r \rangle = 0$. Let ε be an arbitrary positive number. Since $\mathcal{C}$ is dense in $\mathcal{D}$ we can choose a compactly supported function $g \in \mathcal{D}$ such that

$$\|f - g\|_{\mathcal{D}} < \frac{1}{2K} \varepsilon$$

Also we know that $\lim_{r \rightarrow \infty} \langle R_k * g, e_r \rangle = 0$, so we can then find some $\rho > 0$ such that $r > \rho$ implies

$$|\langle R_k * g, e_r \rangle| < \frac{1}{2}\varepsilon$$

Now we can use this and the uniform boundedness property to get

$$\begin{aligned} |\langle R_k * f, e_r \rangle| &\leq |\langle R_k * (f - g), e_r \rangle| + |\langle R_k * g, e_r \rangle| \\ &\leq K \|f - g\|_{\mathcal{D}} + \frac{1}{2}\varepsilon \\ &< \varepsilon \end{aligned}$$

for all $r > \rho$. This completes the proof. ■

The Range of the Radon–John Transform and Its Relation to Partial Differential Equations

4.1 Introduction

“One of the most interesting problems concerning integral transforms is how to characterize the range of certain function and distribution spaces under such transforms” (Gonzalez [223]).

Recall that the Radon–John transform is the same as the k -plane transform. The k -plane transform $P : L^1(\mathbb{R}^n) \rightarrow L^1(\mathcal{G}_{k,n})$ is an injective bounded linear transformation, but it is not surjective. All but the failure of surjectivity was proved in chapter 3, theorem 3.131. If the k -plane transform is not surjective, then what is its range? In this chapter we characterize the range of the k -plane transform acting on various function spaces.

The interest in this question goes back to Radon’s pioneering 1917 paper [508]. We have already seen in earlier chapters that Radon solved the problem of inverting what is now called the Radon transform. In his paper Radon also gave conditions for a function to be in the range of the Radon transform ([508], Satz IV).

Conditions which guarantee that a function g defined on $\mathcal{G}_{k,n}$ is in the range of the k -plane transform are called **consistency conditions** and theorems describing the range are naturally called **range theorems**. In this chapter we explore the range of certain Radon transforms and their associated consistency conditions.

Consistency conditions may include requirements for compact support, the need to satisfy a system of partial differential equations or the need to satisfy a moment condition. A typical moment condition for the Radon transform states that a function g defined on $S^{n-1} \times \mathbb{R}$ must satisfy the following requirements: for every nonnegative integer m the function $\theta \mapsto \int_{\mathbb{R}} t^m g(\theta, t) dt$ is the restriction of a homogeneous polynomial of degree m to the unit sphere S^{n-1} . Analogous moment conditions can be defined for the k plane transform; see theorem 4.3, theorem 4.24, definition 4.26 and theorem 4.27.

The integral $\int_{\mathbb{R}} t^m g(\theta, t) dt$ is in fact the m th moment of the function $g(\theta, t)$ with respect to the variable t and this is the reason for the terminology. This terminology varies considerably among authors. Helgason and Gel’fand were the first authors to

derive range theorems involving moment conditions. Therefore “Gel’fand-Helgason condition” is a common synonym for “moment condition”. Other variants are “Gelfand-Graev-Vilenkin condition”, “Cavalieri condition” or “Helgason – Ludwig condition”.¹ To simplify matters we will refer simply to “moment conditions”.

There has been a long association between Radon transforms and the theory of partial differential equations. This theory is worthy of an entire volume, but because of restrictions in space, we can only present a brief introduction to this topic. This chapter is a good place to present results on differential equations because there is a close connection between solutions of partial differential equations and the range of k -plane transforms with $k < n - 1$. One of the earliest of these results is from John [324] who proved the following theorem for the k -plane transform with $k = 1$, that is, the x-ray transform. In this result a line in $\mathbb{R}^4$ is determined by two distinct points $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{R}^4$, so we can write the x-ray transform as $Xf(\xi, \eta)$. With these coordinates we can state the famous theorem of John:

Theorem 4.1 (John [324]). *A compactly supported C^∞ function v is in the range of the x-ray transform on $C_0^\infty(\mathbb{R}^4)$ if and only if v satisfies the system of partial differential equations:*

$$\left(\frac{\partial^2}{\partial \xi_i \partial \eta_k} - \frac{\partial^2}{\partial \xi_k \partial \eta_i} \right) \left(\frac{v(\xi, \eta)}{|\xi - \eta|} \right) = 0.$$

Furthermore, all C_0^∞ solutions to the ultrahyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0$$

can be found via a simple transformation from functions in the range of the x-ray transform.

This theorem indicates the reciprocal relationship between ranges of k -plane transforms and solutions to partial differential equations. On the one hand the range of a k -plane transform is characterized by functions satisfying a system of partial differential equations, whereas, on the other hand, the solutions to a particular partial differential equation are found by some operation on the range of a k -plane transform.

The prerequisites for reading this chapter include those for chapter 3, and we expect the reader to be at least familiar with the results and notation of chapters 2 and 3. In particular, we make frequent use of the derivative theorem for the Fourier transform (section 3.2) and the generalized polar coordinate theorem (section 3.3.4, theorem 3.9). We also use the well-known fact that the C_0^∞ functions and the rapidly decreasing functions are dense in L^p , $1 \leq p < \infty$.

¹ Smith, Solmon and Wagner [566] mistakenly attributed a range theorem for the Radon transform to Ludwig. However, they published a correction to [566] in which they stated “The theorem characterizing the range of the Radon transform which was credited to D. Ludwig was obtained first by S. Helgason. . .” (Smith, Solmon and Wagner, Addendum to “Practical and Mathematical Aspects of the Problem of Reconstructing Objects from Radiographs”, Bull. Amer. Math. Soc. 84 (1978), no. 4, p. 691.)

Of importance in this chapter are the results on the extension of the k -plane transform to an unbounded operator $\overline{P}$ on L^2 . These results were covered in section 3.9 of chapter 3. In definition 3.87 we defined subspaces of L^2 by

$$\begin{aligned}\mathcal{D}_s(\mathbb{R}^n) &= \left\{ f \in L^2(\mathbb{R}^n) : |\xi|^s \widehat{f}(\xi) \in L^2(\mathbb{R}^n) \right\} \\ \mathcal{D}_s(\mathcal{G}_{k,n}) &= \left\{ g \in L^2(\mathcal{G}_{k,n}) : |\xi''|^s \widehat{g}(\eta, \xi'') \in L^2(\mathcal{G}_{k,n}) \right\}.\end{aligned}$$

The domain of the extended k -plane transform $\overline{P}$ was found to be $\mathcal{D}_{-\frac{k}{2}}$. This extension had the same value as the usual k -plane transform on any function f in $\mathcal{D}_{-\frac{k}{2}}$ such that $(1 + |x|)^{k-n} f(x) \in L^1(\mathbb{R}^n)$, but this operator was also defined by a limiting process for all functions in $\mathcal{D}_{-\frac{k}{2}}$. Corollary 3.99 showed that the range of $\overline{P}$ on $\mathcal{D}_{-\frac{k}{2}} \cap \mathcal{D}_t$ was contained $\mathcal{D}_{t+\frac{k}{2}}(\mathcal{G}_{k,n})$. In particular, the range of $\overline{P}$ on its natural domain $\mathcal{D}_{-\frac{k}{2}}$ is contained in $L^2(\mathcal{G}_{k,n})$.

The extended k -plane transform $\overline{P}$ was shown to have an adjoint $\overline{P}^*$ as an unbounded operator on L^2 . We saw that for any function g in the domain of the adjoint we have $\overline{P}^*g = P^\#g$ and $P^\#g \in L^2(\mathbb{R}^n)$. Therefore, we can denote the adjoint either by $\overline{P}^*g$ or $P^\#g$.

Many of the results in this chapter depend on the extended slice-projection theorem 3.94 and the inversion formula for the extended k -plane transform in theorem 3.106.

The extension of the k -plane transform to an unbounded operator implies the same for the Radon transform. The results are the same, except that we use $S^{n-1} \times \mathbb{R}$ instead of $\mathcal{G}_{n-1,n}$ as the domain of $\overline{R}f$.

We begin the chapter with a more modern version of the range theorem of Radon referred to above. This theorem leads to a set of consistency conditions for the extended k -plane transform. Then we find necessary and sufficient consistency conditions for a function to be the Radon transform of a compactly supported L^2 function. This result is used to prove a version of the Paley–Wiener theorem for the Fourier transform. Then we use this Paley–Wiener theorem to get necessary and sufficient consistency conditions for a function to be in the range of the k -plane transform acting on certain compactly supported functions. Most of these results, in the L^2 case, were first proved by Smith, Solmon, and Wagner [566] and Solmon [569]. The proofs here are according to these authors, but we present a few more details.

We then present range results for smooth functions. The pioneers in this study are John [322], Helgason [267, 269], and Ludwig [405]. This discussion is brief and most of the proofs are omitted.

Finally, we present a brief introduction to the relation between the Radon transform and partial differential equations.

4.2 The Range of the k -Plane Transform on Compactly Supported L^2 Functions

We begin with a simple range theorem for the Radon transform. Then we give consistency conditions sufficient for a function to be in the range of the k -plane transform. This is followed by necessary and sufficient conditions for a function to be the image

under the Radon transform of a compactly supported function. This result leads to an interesting version of the Paley–Wiener theorem.

The Paley–Wiener theorem gives range and consistency conditions for the Fourier transform. It is usually proved directly by using Fourier transform methods. It is interesting that we can prove an n -dimensional version of the Paley–Wiener theorem by using only the relatively simple one-dimensional Paley–Wiener theorem and the range results of this chapter. The Paley–Wiener theorem is then used to derive the range of the k -plane transform acting on compactly supported functions.

The following simple range theorem for the Radon transform is from Smith, Solmon, and Wagner [566]. This is the more modern version of the range theorem of Radon referred to in the introduction. However, there is no such simple range theorem for the k -plane transform when $k < n - 1$.

Theorem 4.2. *The range of $\bar{R}$ on $\mathcal{D}_s(\mathbb{R}^n) \cap \mathcal{D}_{\frac{1-n}{2}}(\mathbb{R}^n)$ consists of the even functions in $\mathcal{D}_{\frac{n-1}{2}+s}(S^{n-1} \times \mathbb{R}) \cap \mathcal{D}_{\frac{n-1}{2}}(S^{n-1} \times \mathbb{R})$. In particular, the range of the extended Radon transform on $\mathcal{D}_{\frac{1-n}{2}}(\mathbb{R}^n)$ consists of the even functions in $\mathcal{D}_{\frac{n-1}{2}}(S^{n-1} \times \mathbb{R})$.*

Proof. Let $g = \bar{R}f$, where $f \in \mathcal{D}_s \cap \mathcal{D}_{\frac{1-n}{2}}$. Since $f \in \mathcal{D}_s$, the isometry theorem 3.97 shows that $g \in \mathcal{D}_{\frac{n-1}{2}+s}$, but f is also square integrable and thus in $\mathcal{D}_t$ with $t = 0$. The isometry theorem in this case implies that $g \in \mathcal{D}_{\frac{n-1}{2}}$. Because we know that the Radon transform generates an even function, this shows that the range of $\bar{R}$ on $\mathcal{D}_s \cap \mathcal{D}_{\frac{1-n}{2}}$ is a subset of the even functions in $\mathcal{D}_{\frac{n-1}{2}+s} \cap \mathcal{D}_{\frac{n-1}{2}}$.

Conversely, if g is an even function in $\mathcal{D}_{\frac{n-1}{2}+s} \cap \mathcal{D}_{\frac{n-1}{2}}$, then define $F(\tau\theta) = (2\pi)^{\frac{1-n}{2}} \widehat{g}(\theta, \tau)$. The function F is well defined almost everywhere on $\mathbb{R}^n$, because any nonzero point of $\mathbb{R}^n$ is representable in the form $\tau\theta$ for only two pair of points in $S^{n-1} \times \mathbb{R}$, namely, (θ, τ) and $(-\theta, -\tau)$, where $\theta = \frac{x}{|x|}$. Because g is even, the two possible values for F are actually the same. Using polar coordinates we have

$$\int_{\mathbb{R}^n} |\xi|^{2\alpha} |F(\xi)|^2 d\xi = \frac{1}{2} (2\pi)^{1-n} \int_{S^{n-1}} \int_{-\infty}^{\infty} |\tau|^{n-1+2\alpha} |\widehat{g}(\theta, \tau)|^2 d\tau d\theta. \quad (4.1)$$

If we set $\alpha = 0$, then the right-hand side is finite because $g \in \mathcal{D}_{\frac{n-1}{2}}$. Therefore, $F \in L^2(\mathbb{R}^n)$ and hence F has an inverse Fourier transform which we denote by f .

We now show that $f \in \mathcal{D}_s \cap \mathcal{D}_{\frac{1-n}{2}}$. If we set $\alpha = \frac{1-n}{2}$, in equation (4.1), then the right-hand side is finite because $g \in L^2$. The finiteness of the left-hand integral then shows that $f \in \mathcal{D}_{\frac{1-n}{2}}$. A similar consideration with $\alpha = s$, and $g \in \mathcal{D}_{\frac{n-1}{2}+s}$ shows that $f \in \mathcal{D}_s$.

It only remains to show that $\bar{R}f = g$. Since $f \in \mathcal{D}_{\frac{1-n}{2}}$, then it is in the domain of the extended Radon transform $\bar{R}$. By the extended slice-projection theorem we see that for almost every $\tau \in \mathbb{R}$ and almost every direction θ we have

$$\begin{aligned} \widehat{\bar{R}f}(\theta, \tau) &= (2\pi)^{\frac{n-1}{2}} \widehat{f}(\tau\theta) \\ &= (2\pi)^{\frac{n-1}{2}} F(\tau\theta) \\ &= \widehat{g}(\theta, \tau) \end{aligned}$$

Fourier inversion implies that $\bar{R}f = g$. ■

Theorem 4.2 is fairly general. For example, given an even function g in $\mathcal{D}_{\frac{n-1}{2}}(S^{n-1} \times \mathbb{R})$, theorem 4.2 guarantees the existence of a function f in $\mathcal{D}_{\frac{1-n}{2}}(\mathbb{R}^n)$ such that $\overline{R}f = g$. If g is rapidly decreasing or compactly supported, then there is no guarantee that f would be rapidly decreasing or compactly supported; Smith, Solmon and Wagner have an example of a C^∞ function f such that $\overline{R}f$ has compact support whereas f does not (see [566], page 1263.) If such a condition is desired, then more restrictions have to be placed on g . Typically such restrictions require g to satisfy certain moment conditions or they require that g satisfy a system of partial differential equations. Furthermore these restrictions are required for characterizations of the range of the k plane transform when $k < n - 1$. We already observed this phenomenon in John's theorem 4.1.

A moment condition for the Radon transform states that the function $q(\theta) = \int_{\mathbb{R}} t^m g(\theta, t) dt$ is the restriction of a homogeneous polynomial to the unit sphere S^{n-1} . Analogous moment conditions can be defined for the k plane transform; see condition 3 of the next theorem, theorem 4.24, definition 4.26 and theorem 4.27.

The next theorem gives sufficient conditions for a function to be in the range of the k -plane transform. We will prove later, in theorem 4.24, that these conditions are *necessary and sufficient* for a *compactly supported function* f to exist in $\mathcal{D}_s$ such that $g = \overline{P}f$.

Theorem 4.3. *Let $\overline{P}$ be the extended k -plane transform and let g be a measurable function on $\mathcal{G}_{k,n}$ which satisfies the following three conditions:*

1. $g \in \mathcal{D}_{\frac{k}{2}+s}$,
2. *For almost every $\eta \in G_{k,n}$ the function $x'' \mapsto g(\eta, x'')$ has compact support on $\eta^\perp$, and*
3. *For each nonnegative integer m , a homogeneous polynomial Q_m of degree m exists on $\mathbb{R}^n$ such that the restriction of Q_m to any $n - k$ plane $\eta^\perp$ satisfies*

$$Q_m(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''$$

for $\xi'' \in \eta^\perp$.

Then there exists some $f \in \mathcal{D}_s \cap \mathcal{D}_{-\frac{k}{2}}$ such that $\widehat{f}$ is analytic on $\mathbb{R}^n$ and such that $g = \overline{P}f$.

Remark 4.4. Although each integral $\int_{\eta^\perp} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''$ is a homogeneous polynomial of degree m on the $n - k$ plane $\eta^\perp$, there is no a priori guarantee that they “mesh” together to form a global homogeneous polynomial on $\mathbb{R}^n$. It is this meshing that is the importance of consistency condition 3.

We need the following lemma before proceeding to the proof.

Lemma 4.5. *Let $f \in L^1_0(\mathbb{R}^n)$. Define functions $Q_{m,f}$, one for each nonnegative integer m , by*

$$Q_{m,f}(\xi) = \int_{\mathbb{R}^n} \langle \xi, x \rangle^m f(x) dx. \quad (4.2)$$

Then the function $Q_{m,f}$ is a homogeneous polynomial of degree m on $\mathbb{R}^n$.

Furthermore, for $\zeta \in \mathbb{C}^n$, the function $(2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_{m,f}(\zeta)$ is an entire analytic function whose restriction to $\mathbb{R}^n$ satisfies

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_{m,f}(\xi). \quad (4.3)$$

Finally, if P is the k -plane transform on $\mathbb{R}^n$, then the homogeneous polynomials $Q_{m,f}$ on $\mathbb{R}^n$ are such that their restrictions to any $n-k$ plane $\eta^\perp$ satisfy

$$Q_{m,f}(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m P f(\eta, x'') dx'' \quad (4.4)$$

for $\xi'' \in \eta^\perp$.

Remark 4.6. A careful reading of the proof will show that we can replace $\mathbb{R}^n$ by any k -space η in $\mathbb{R}^n$ as long as the Fourier transform is taken to be the k -dimensional Fourier transform on η . In that case the constant $(2\pi)^{-\frac{n}{2}}$ must of course be changed to $(2\pi)^{-\frac{k}{2}}$.

Proof. Since f is a compactly supported L^1 function, then the same is true for $x^\alpha f(x)$ for any multi-index α . It is then clear that $\int_{\mathbb{R}^n} \langle \xi, x \rangle^m f(x) dx$ expands into a finite linear combination with each coefficient being an integral of some $x^\alpha f(x)$, times powers of the form ξ^α with $|\alpha| = m$. Because $f \in L^1$, then these coefficients are finite. Hence, $Q_{m,f}(\xi)$ is a homogeneous polynomial of degree m on $\mathbb{R}^n$.

Before proving equation (4.3), let us consider, for fixed $\xi \in \mathbb{R}^n$, the function $G_\xi(x) = \sum_{m=0}^{\infty} \frac{1}{m!} |f(x)| \langle |\xi|, |x| \rangle^m$. By the Cauchy-Schwarz inequality, the series defining this function is dominated by the series for $|f(x)| e^{|\xi||x|}$. Therefore the series defining G_ξ converges pointwise. Furthermore, each term of the series defining G_ξ is integrable. This is because f is compactly supported so there exists a constant C such that the support of f is contained in the ball of radius C . Consequently $\int_{\mathbb{R}^n} |f(x)| \langle |\xi|, |x| \rangle^m dx \leq |C\xi|^m \|f\|_{L^1}$. The terms of the series $\sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{R}^n} |f(x)| \langle |\xi|, |x| \rangle^m dx$ are dominated by the terms of the series $\|f\|_{L^1} \sum_{m=0}^{\infty} \frac{|C\xi|^m}{m!} = e^{C|\xi|} \|f\|_{L^1} < \infty$. The monotone convergence theorem now shows that the function G_ξ is integrable and that its integral can be computed by term by term integration of its defining series. Now we can use G_ξ to apply Lebesgue's dominated convergence theorem to conclude that the series $\sum_{m=0}^{\infty} \frac{1}{m!} f(x) \langle \xi, x \rangle^m$ is defined almost everywhere and is an integrable function such that

$$\sum_{m=0}^{\infty} \frac{i^{-m}}{m!} \int_{\mathbb{R}^n} f(x) \langle \xi, x \rangle^m dx = \int_{\mathbb{R}^n} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} f(x) \langle \xi, x \rangle^m dx. \quad (4.5)$$

Since $Q_{m,f}(\xi) = \int_{\mathbb{R}^n} \langle \xi, x \rangle^m f(x) dx$, then the power series on the right-hand side of equation (4.3) converges for all $\xi \in \mathbb{R}^n$ and thus extends to an entire analytic function on $\mathbb{C}^n$. We next show that the restriction of this analytic function to $\mathbb{R}^n$ equals $\widehat{f}$.

Because f is L^1 , then

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} dx$$

for $\xi \in \mathbb{R}^n$. Replacing replacing $e^{-i\langle \xi, x \rangle}$ by its Taylor series expansion leads to $\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \sum_{m=0}^{\infty} \frac{1}{m!} (-i\langle \xi, x \rangle)^m dx$. Then equation (4.5) gives

$$\begin{aligned} \widehat{f}(\xi) &= (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} \int_{\mathbb{R}^n} f(x) \langle \xi, x \rangle^m dx \\ &= (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_{m,f}(\xi) \end{aligned} \quad (4.6)$$

and this establishes equation (4.3).

By proposition 3.24 (chapter 3) with $\rho(t) = t^m$ we get

$$\int_{\mathbb{R}^n} \langle \xi'', x \rangle^m f(x) dx = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m Pf(\eta, x'') dx''.$$

From the definition of $Q_{m,f}$ we thus have the conclusion (4.4). ■

Proof of theorem 4.3. Assuming that conditions 1–3 hold, define $g_\eta(x'') = g(\eta, x'')$ and define $q_{m,g,\eta}(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''$.

We begin by showing that g_η is in $L^2(\eta^\perp)$ for almost every $\eta \in G_{k,n}$. Since $g \in \mathcal{D}_{\frac{k}{2}+s}(\mathcal{G}_{k,n})$ then in particular $g \in L^2(\mathcal{G}_{k,n})$. Hence $\int_{G_{k,n}} \int_{\eta^\perp} |g(\eta, x'')|^2 dx'' d\eta$ is finite and Fubini implies that g_η is in $L^2(\eta^\perp)$ for almost every $\eta \in G_{k,n}$. Condition 2 means that g_η is compactly supported on $\eta^\perp$. Therefore, g_η is in $L^1(\eta^\perp)$ and hence its $(n-k)$ -dimensional Fourier transform $\widehat{g}_\eta$ on the $(n-k)$ -plane $\eta^\perp$ is well defined. Furthermore, we can use remark 4.6 to apply lemma 4.5 on the $(n-k)$ -plane $\eta^\perp$, thus giving

$$\widehat{g}_\eta(\xi'') = (2\pi)^{-\frac{n-k}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} q_{m,g,\eta}(\xi''). \quad (4.7)$$

By condition 3, there exist homogeneous polynomials Q_m of degree m on $\mathbb{R}^n$ such that

$$Q_m(\xi'') = q_{m,g,\eta}(\xi'') \quad (4.8)$$

Equations (4.7) and (4.8) show that for almost every η and every $\xi'' \in \eta^\perp$ the series $\sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\xi'')$ is convergent. But any $\xi \in \mathbb{R}^n$ is in some $\eta^\perp$ so $\sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\xi)$ is convergent for all $\xi \in \mathbb{R}^n$. Furthermore, if $\xi \in \mathbb{R}^n$ is in two different $(n-k)$ -planes $\eta^\perp$, then the same series results.

Therefore, we can define $F(\xi) = (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\xi)$ and F will be well defined and convergent everywhere on $\mathbb{R}^n$. In particular, F is analytic on $\mathbb{R}^n$. Furthermore, equations (4.7) and (4.8) show that

$$F(\xi'') = (2\pi)^{-\frac{k}{2}} \widehat{g}_\eta(\xi''). \quad (4.9)$$

for $\xi'' \in \eta^\perp$.

We proceed to show that $F \in L^2(\mathbb{R}^n)$. By equation (4.9) and the generalized polar coordinate theorem we have

$$\begin{aligned} \|F\|_{L^2(\mathbb{R}^n)}^2 &= c \int \int_{G_{k,n} \eta^\perp} |\xi''|^k |F(\xi'')|^2 d\xi'' \\ &= c \int \int_{G_{k,n} \eta^\perp} |\xi''|^k |\widehat{g}(\eta, \xi'')|^2 d\xi'' < \infty, \end{aligned} \quad (4.10)$$

since by condition 1 we have $g \in \mathcal{D}_{\frac{k}{2}}$.

Hence F is L^2 and therefore has an inverse Fourier transform which we denote by f . We now show that $f \in \mathcal{D}_s$.

Because $f \in L^2$, it is enough to show that $\int_{\mathbb{R}^n} \|\xi\|^s |\widehat{f}(\xi)|^2 d\xi$ is finite. But $\widehat{f} = F$ and we have

$$\begin{aligned} \int_{\mathbb{R}^n} \|\xi\|^s |\widehat{f}(\xi)|^2 d\xi &= c \int \int_{G_{k,n} \eta^\perp} |\xi''|^k \|\xi''\|^s |F(\xi'')|^2 d\xi'' \\ &= c \int \int_{G_{k,n} \eta^\perp} \left| |\xi''|^{\frac{k}{2}+s} \widehat{g}(\eta, \xi'') \right|^2 d\xi'' < \infty \end{aligned}$$

The last term is finite as a consequence of condition 1.

The same calculation using $s = -\frac{k}{2}$ and the fact that from condition 1 the function $\widehat{g}(\eta, \xi'')$ is also L^2 shows that $f \in \mathcal{D}_{-\frac{k}{2}}$. Hence, f is in the domain of the extended k -plane transform. Also this allows us to use the extended slice-projection theorem 3.94 and equation (4.9) to get

$$\begin{aligned} \widehat{\overline{P}f}(\eta, \xi'') &= (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'') = \\ (2\pi)^{\frac{k}{2}} F(\xi'') &= \widehat{g}_\eta(\xi'') = \widehat{g}(\eta, \xi'') \end{aligned}$$

almost everywhere. It follows by Fourier inversion that $\overline{P}f = g$. Because we already observed that $\widehat{f}$ is analytic and that f is in $\mathcal{D}_s \cap \mathcal{D}_{-\frac{k}{2}}$, we are finished. ■

From hypothesis 2 of this theorem, we might expect that there would be a compact set K such that the function f would have support on the set K . This in fact turns out to be essentially true, but it is not an immediate consequence of theorem 4.3; see the discussion following the proof of theorem 4.2. We will see later that the consistency conditions 2 and 3 are crucial for demonstrating that if Pf has compact support then so does f ; see theorems 4.12, 4.24 and 4.27. However, the only use of the consistency conditions in theorem 4.3 was to guarantee that the function F was well defined.

We next have the following range result for “radial” type compactly supported functions on $S^{n-1} \times \mathbb{R}$.

Theorem 4.7. *Let $\Phi(\theta, s) = \varphi(s)$, where φ is an even function in $C_0^\infty[-1, 1]$. Then Φ is in the range of the Radon transform, and furthermore $\Phi = R\psi$, where $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\psi(x) = 0$ for $|x| > 1$.*

Proof. We first deal with the case where n is odd.

Since Φ is even and in $C_0^\infty(S^{n-1} \times \mathbb{R})$, then it is even and in $\mathcal{D}_{n-1} \cap \mathcal{D}_{\frac{n-1}{2}}$ on $S^{n-1} \times \mathbb{R}$ (compare, theorem 3.89). Therefore, theorem 4.2 applies with $s = \frac{n-1}{2}$ and therefore there exists a function $\psi \in \mathcal{D}_{\frac{n-1}{2}} \cap \mathcal{D}_{\frac{1-n}{2}}$ on $\mathbb{R}^n$ such that $\Phi = \bar{R}\psi$. These properties of ψ show that we can apply conclusion 2 of inversion theorem 3.106 in chapter 3 with $s = k = n - 1$ to obtain

$$\psi = c R^\# \Lambda^{n-1} \bar{R} \psi = c R^\# \Lambda^{n-1} \Phi$$

Since n is odd, $|t|^{n-1} = t^{n-1}$, and an easy calculation using the derivative theorem for the Fourier transform shows that $\Lambda^{n-1} \Phi = (-i)^{n-1} \frac{d^{n-1} \Phi}{dt^{n-1}}$. Hence, $\psi = c R^\# \varphi^{(n-1)}$. By the definition of the backprojection operator we then get

$$\psi(x) = c \int_{S^{n-1}} \varphi^{(n-1)}(\langle x, \theta \rangle) d\theta.$$

An easy induction argument shows that any derivative with respect to x of the integrand is a finite linear combination of the form $\varphi^{(j)}(\langle x, \theta \rangle) x^\beta \theta^\gamma$. Since $\varphi^{(n-1)}$ is a C^∞ function, then each of these derivatives is integrable on the sphere, and thus we can compute any derivative $\partial^\alpha \psi$ by differentiating under the integral sign. We are not interested in the actual expression for the derivatives, but this argument shows that $\psi \in C^\infty(\mathbb{R}^n)$.

Because Φ is radial, we can obtain another expression for ψ by using theorem 3.31 to evaluate the backprojection:

$$\begin{aligned} \psi(x) &= c \frac{1}{|x|^{n-2}} \int_0^{|x|} \varphi^{(n-1)}(s) \left(|x|^2 - s^2 \right)^{\frac{(n-3)}{2}} ds \\ &= c \frac{1}{|x|} \int_{-|x|}^{|x|} \varphi^{(n-1)}(s) \left(1 - \frac{s^2}{|x|^2} \right)^{\frac{(n-3)}{2}} ds \end{aligned} \quad (4.11)$$

for some constant c . The last transformation is valid because, n being odd, $\varphi^{(n-1)}$ is an even function. We can now use this equation to show that ψ is supported in the unit ball. When dealing with the Radon transform, the minimum odd value of the dimension n is 3. Then $\frac{(n-3)}{2}$ is a nonnegative integer and hence $\left(1 - \frac{s^2}{|x|^2}\right)^{\frac{(n-3)}{2}}$ is a polynomial in s of degree $n - 3$. Therefore, if we perform $n - 2$ integrations by parts in equation (4.11), then the integrand is 0 and what remains is a linear combination of terms of the form $\varphi^{(j)}(|x|)$. But the hypothesis implies that any such derivative is zero if $|x| > 1$. As a consequence, $\psi(x) = 0$ for $|x| > 1$. This completes the proof in the case that n is odd.

If n is even define Φ_1 on $S^n \times \mathbb{R}$ by $\Phi_1(\vartheta, s) = \varphi(s)$. As shown in the first part of the proof, it follows that Φ_1 is in the range of the $n + 1$ -dimensional Radon transform, because $n + 1$ is odd. Let $\psi_1 \in C_0^\infty(\mathbb{R}^{n+1})$ be such that the $n + 1$ -dimensional Radon transform of ψ_1 is Φ_1 . Then a straightforward calculation shows that if we define

$\psi(x) = \int_{-\infty}^{\infty} \psi_1(x, x_{n+1}) dx_{n+1}$ for $x \in \mathbb{R}^n$, then the n -dimensional Radon transform of ψ is Φ and ψ satisfies the other conditions of the theorem. ■

We require the following three lemmas.

Lemma 4.8. *If F is an L^2 function on $\mathbb{R}$ which is orthogonal to every C^∞ function with support in the interval $[-1, 1]$, then f has support in the complement of $[-1, 1]$.*

Also, if F is an even L^2 function on $\mathbb{R}$ which is orthogonal to every even C^∞ function with support in the interval $[-1, 1]$, then f has support in the complement of $[-1, 1]$.

Proof. We do the proof for the case of even functions, the more general case is similar. Let $\mathcal{E}_1$ be the Hilbert space of even square integrable functions with support in $[-1, 1]$ and let $\mathcal{E}_2$ be the Hilbert space of even square integrable functions with support outside $[-1, 1]$. Then $\mathcal{E}_1 + \mathcal{E}_2$ equals the Hilbert space of all even square integrable functions defined on $\mathbb{R}$. Because F is such a function we can write $F = F_1 + F_2$, where $F_1 \in \mathcal{E}_1$, $F_2 \in \mathcal{E}_2$, and $F_1 \perp F_2$. From this we see that

$$\|F_1\|_{L^2}^2 = \langle F_1, F_1 \rangle = \langle F_1, F_1 \rangle + \langle F_1, F_2 \rangle = \langle F_1, F \rangle.$$

We now choose a sequence (ρ_j) of even C^∞ functions with support in the interval $[-1, 1]$, such that $F_1 = \lim_{j \rightarrow \infty} \rho_j$. Then each such ρ_j is in $\mathcal{E}_1$ and we know that F is orthogonal to each such ρ_j . Consequently, $\|F_1\|_{L^2}^2 = \lim_{j \rightarrow \infty} \langle \rho_j, F \rangle = 0$. This proves that $F_1 = 0$ and hence $F = F_2$, which vanishes almost everywhere on $[-1, 1]$. ■

Lemma 4.9. *If $\varphi \in C_0^\infty$ and $\varphi(x) = 0$ for $|x| < 1$, then*

$$\int_{S^{n-1}} \Lambda^{n-1} R\varphi(\theta, t) d\theta = 0 \text{ for } |t| \leq 1.$$

Proof. If a function $\gamma(\theta, t)$ defined on $S^{n-1} \times \mathbb{R}$ is a function only of t , then we abuse the language by using the same symbol γ for the function $\gamma(\theta, t)$ defined on $S^{n-1} \times \mathbb{R}$ and the associated function $\gamma(t)$ defined on $\mathbb{R}$.

If we let $F(t) = \int_{S^{n-1}} \Lambda^{n-1} R\varphi(\theta, t) d\theta$, then all we need to show is that F is supported outside the interval $[-1, 1]$. It is obvious that $\langle F(t), \rho(t) \rangle_{\mathbb{R}} = \langle \Lambda^{n-1} R\varphi, \rho \rangle_{S^{n-1} \times \mathbb{R}}$.

If we now let $\rho = \rho(t)$ be an arbitrary even function in $C_0^\infty[-1, 1]$, then by theorem 4.7 there is a function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(x) = 0$ for $|x| > 1$ such that $R\psi = \rho$. Using the inversion formula for the extended Radon transform we get

$$\begin{aligned} \langle \Lambda^{n-1} R\varphi, \rho \rangle &= \langle \Lambda^{n-1} R\varphi, R\psi \rangle \\ &= \langle R^\# \Lambda^{n-1} R\varphi, \psi \rangle = 2(2\pi)^{n-1} \langle \varphi, \psi \rangle \end{aligned}$$

Note that this is a valid computation because conclusion 4, theorem 3.106, implies that $\Lambda^{n-1} R\varphi$ is in the domain of R^* . But $\langle \varphi, \psi \rangle = 0$ because $\varphi(x) = 0$ for $|x| < 1$, whereas $\psi(x) = 0$ for $|x| > 1$. Combining these results we obtain $\langle F(t), \rho(t) \rangle_{\mathbb{R}} = 0$

for any even function ρ in $C_0^\infty[-1, 1]$. Furthermore, because $R\varphi$ is an even function, it is easy to check that F is also even. By lemma 4.8 F must vanish almost everywhere in $[-1, 1]$. ■

Lemma 4.10. *If $\varphi \in C_0^\infty$, $\varphi(x) = 0$ for $|x| < 1$ and q is a polynomial of degree m , then*

$$\int_{S^{n-1}} q(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta \quad (4.12)$$

is a polynomial in t of degree $< m$ on the subset of $\mathbb{R}$ where $|t| < 1$.

Proof. If q is not homogeneous, we can break up the given integral into a finite sum in which the q for each term is homogeneous. Therefore, there is no loss of generality by assuming that q is a homogeneous polynomial.

With the hypothesis on φ there is no problem moving derivatives into the integral (4.12). Also, an easy observation from the derivative theorem for the Fourier transform is that Λ^{n-1} commutes with differential operators, so we therefore obtain

$$\frac{\partial^m}{\partial t^m} \int_{S^{n-1}} q(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta = \int_{S^{n-1}} \Lambda^{n-1} q(\theta) \frac{\partial^m}{\partial t^m} R\varphi(\theta, t) d\theta. \quad (4.13)$$

Letting $q(\partial)$ be the differential operator associated to the polynomial q , we can show that

$$q(\theta) \frac{\partial^m}{\partial t^m} R\varphi(\theta, t) = R(q(\partial)\varphi)(\theta, t) \quad (4.14)$$

In fact, this follows directly from the next calculation using the homogeneity of q , the slice-projection theorem and the derivative theorem for the Fourier transform

$$\begin{aligned} (R(q(\partial)\varphi))^\wedge(\theta, \tau) &= (2\pi)^{\frac{n-1}{2}} (q(\partial)\varphi)^\wedge(\tau\theta) \\ &= (2\pi)^{\frac{n-1}{2}} q(i\tau\theta) \widehat{\varphi}(\tau\theta) = q(\theta) (i\tau)^m \widehat{R\varphi}(\theta, \tau) \\ &= \left(q(\theta) \frac{\partial^m}{\partial t^m} R\varphi \right)^\wedge(\theta, \tau). \end{aligned}$$

If we now combine equations (4.13) and (4.14), we arrive at

$$\frac{\partial^m}{\partial t^m} \int_{S^{n-1}} q(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta = \int_{S^{n-1}} \Lambda^{n-1} R(q(\partial)\varphi)(\theta, t) d\theta.$$

But $q(\partial)\varphi$ satisfies the hypotheses of lemma 4.9, so we can conclude that

$$\frac{\partial^m}{\partial t^m} \int_{S^{n-1}} q(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta = 0$$

for $|t| < 1$. Therefore, the function $t \mapsto \int_{S^{n-1}} q(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta$ has a vanishing derivative of order m on the set where $|t| < 1$, so this function must be a polynomial of degree $< m$ on this set. ■

Definition 4.11. If K is a compact subset of $\mathbb{R}^n$, we define its *support function* to be the function of $\xi \in \mathbb{R}^n$ defined by

$$\|\xi\|_K = \sup_{x \in K} \langle x, \xi \rangle.$$

It is an easy exercise to show that if K is the rectangle:

$$K = \{x \in \mathbb{R}^n : |x_j| \leq a_j\},$$

then $\|\xi\|_K = \sum a_j |\xi_j|$, and if K is a ball centered at the origin, then $\|\xi\|_K = \frac{1}{2} \text{diam}(K) |\xi|$.

Theorem 4.12, which follows, gives sufficient conditions for a function g defined on $S^{n-1} \times \mathbb{R}$ to be the Radon transform of a function supported on a convex, compact subset of $\mathbb{R}^n$. It resembles theorem 4.3, which gives sufficient conditions for a function to be the k -plane transform of a possibly noncompactly supported function f . Theorem 4.12 yields a compactly supported f but only applies to the Radon transform. Theorem 4.12 is closely related to a theorem of Helgason [267, 269] (compare, Ludwig [405]); the conditions of theorem 4.12 are not only sufficient but also are necessary for g to be the Radon transform of a compactly supported function. Helgason's result applies to smooth functions (class $\mathcal{S}$ or C_0^∞). The proof given here is from Smith, Solmon, and Wagner [566] and applies to functions with less smoothness. Later, in theorem 4.24, we prove that the consistency conditions of theorem 4.12 are necessary and sufficient.

Theorem 4.12. *Let K be a convex, compact subset of $\mathbb{R}^n$, let s be a real number, and let g be a measurable function on $L^2(S^{n-1} \times \mathbb{R})$. Then sufficient conditions for the existence of a function $f \in \mathcal{D}_s$ with support in K such that $g = \bar{R}f$ are:*

1. g is an even function in $\mathcal{D}_{\frac{n-1}{2}+s}$, and
2. $g(\theta, t) = 0$ for $t > \|\theta\|_K$, and
3. for each nonnegative integer m , there exists a homogeneous polynomial p_m of degree m on $\mathbb{R}^n$ such that the restriction of p_m to the unit sphere satisfies $p_m(\theta) = \int_{-\infty}^{\infty} t^m g(\theta, t) dt$.

Proof. We first prove the theorem in the special case where K is the closed unit ball $\bar{B}(1)$.

Let g be a given function which satisfies conditions 1–3. Then g is square integrable on $S^{n-1} \times \mathbb{R}$ and for almost every $\theta \in S^{n-1}$ we have $\int_{\mathbb{R}} |\tau|^{n-1} |\widehat{g}(\theta, \tau)|^2 d\xi'' < \infty$, because $n-1 > 0$ and by this and condition 2 the integrand is a compactly supported continuous function. Hence, $g \in \mathcal{D}_{\frac{n-1}{2}}$ and, furthermore, by condition 1, this implies that g is an even function in $\mathcal{D}_{\frac{n-1}{2}+s} \cap \mathcal{D}_{\frac{n-1}{2}}$. By theorem 4.2, there exists a function $f \in \mathcal{D}_s \cap \mathcal{D}_{\frac{1-n}{2}}$ such that $\bar{R}f = g$.

It remains to show that f has support in the closed-unit ball. To do this we first prove that for any rapidly decreasing function φ we have

$$\langle f, \varphi \rangle_{L^2(\mathbb{R}^n)} = c \left\langle g, \Lambda^{n-1} R\varphi \right\rangle_{L^2(S^{n-1} \times \mathbb{R})} \quad (4.15)$$

The proof of equation (4.15) begins with the observation that since $\varphi \in \mathcal{S}$, then we can use conclusion 4 of theorem 3.106 to show that $\Lambda^{n-1} R\varphi$ is in the domain of the adjoint $\bar{R}^*$ and that $\varphi = c \bar{R}^* \Lambda^{n-1} \bar{R}\varphi$. Hence, $\langle f, \varphi \rangle_{L^2(\mathbb{R}^n)} = c \langle f, \bar{R}^* \Lambda^{n-1} \bar{R}\varphi \rangle_{L^2(\mathbb{R}^n)} = c \langle \bar{R}f, \Lambda^{n-1} \bar{R}\varphi \rangle_{L^2(S^{n-1} \times \mathbb{R})} = c \langle g, \Lambda^{n-1} R\varphi \rangle_{L^2(S^{n-1} \times \mathbb{R})}$. This proves equation (4.15).

At this point we expand g into a Legendre series. The Gram–Schmidt orthonormalization process applied to the restrictions of the monomials t^j to the interval $[-1, 1]$ gives rise to a complete orthonormal basis for the Hilbert space $L^2([-1, 1])$. This basis consists of polynomials L_j of degree j that are orthogonal to all polynomials of degree $< j$. The polynomials L_j are called the **Legendre polynomials**.

Any function h in $L^2([-1, 1])$ can therefore be expanded in a series: $h(t) = \sum_{j=0}^{\infty} c_j L_j(t)$, where L_j is the j th Legendre polynomial and $c_j = \langle h, L_j \rangle_{L^2([-1, 1])} = \int_{-1}^1 h(t) L_j(t) dt$. Such a series is called a **Legendre series**.

We can extend the idea to functions of the form $h(\theta, t)$ by obtaining a Legendre series for each fixed $\theta \in S^{n-1}$.

If $K = \overline{B(1)}$, then $\|\theta\|_K = 1$ for any θ in the unit sphere. Therefore, by condition 2 we see that $g(\theta, t)$ vanishes for $t \notin [-1, 1]$. By condition 1 we have $g \in L^2(S^{n-1} \times [-1, 1])$. These facts imply that g has the Legendre series $g(\theta, t) = \sum_{j=0}^{\infty} c_j(\theta) L_j(t)$, where $c_j(\theta) = \int_{-1}^1 g(\theta, t) L_j(t) dt$. Because L_j is a polynomial of degree j we see that $c_j(\theta)$ can be expressed as a linear combination of integrals of the form $\int_{-1}^1 g(\theta, t) t^k dt$, where $k \leq j$. By this and condition 3, we see that $c_j(\theta)$ is a polynomial of degree j .

Fix θ and let $g_\theta(t) = g(\theta, t)$ and $R_\theta \varphi(t) = R\varphi(\theta, t)$. Then for almost all θ , the functions g_θ and $\Lambda^{n-1} R_\theta \varphi$ are square integrable and, hence, we can expand the following inner product using the Legendre series for g :

$$\begin{aligned} \left\langle g_\theta, \Lambda^{n-1} R_\theta \varphi \right\rangle_{L^2([-1, 1])} &= \sum_{j=0}^{\infty} c_j(\theta) \left\langle L_j, \Lambda^{n-1} R_\theta \varphi \right\rangle \\ &= \sum_{j=0}^{\infty} c_j(\theta) \int_{-1}^1 L_j(t) \Lambda^{n-1} R_\theta \varphi(\theta, t) dt. \end{aligned}$$

If we integrate the last equation over the sphere then we obtain

$$\begin{aligned} \left\langle g, \Lambda^{n-1} R\varphi \right\rangle_{L^2(S^{n-1} \times \mathbb{R})} &= \int_{S^{n-1}} \sum_{j=0}^{\infty} c_j(\theta) \int_{-1}^1 L_j(t) \Lambda^{n-1} R_\theta \varphi(t) dt d\theta \\ &= \sum_{j=0}^{\infty} \int_{-1}^1 L_j(t) \left(\int_{S^{n-1}} c_j(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta \right) dt. \end{aligned}$$

If we now restrict attention to those rapidly decreasing functions φ , which are compactly supported C^∞ functions with support outside $[-1, 1]$, then since $c_j(\theta)$ is a polynomial of degree j , we can apply lemma 4.10 to conclude that $\int_{S^{n-1}} c_j(\theta) \Lambda^{n-1} R\varphi(\theta, t) d\theta$ is a polynomial of degree $< j$ on $[-1, 1]$. Because $L_j(t)$

is orthogonal to such polynomials we get $\langle g, \Lambda^{n-1} R\varphi \rangle_{L^2(S^{n-1} \times \mathbb{R})} = 0$. Combining this result with equation (4.15) shows that $\langle f, \varphi \rangle_{L^2(\mathbb{R}^n)} = 0$ for all C_0^∞ functions φ which vanish for $|x| < 1$. By lemma 4.8 f must vanish almost everywhere outside $[-1, 1]$.

This completes the demonstration that the theorem is true if K is the closed-unit ball. We leave without proof the relatively simple extension to the case where K is a closed ball of any center and radius.

Finally, we can establish the general case of the theorem. Let g be a given function that satisfies conditions 1–3 for a given compact convex set K . Let B be a ball containing K . Conditions 1 and 3 are independent of K and thus are satisfied for B . Condition 2 is satisfied because $\|\xi\|_K \leq \|\xi\|_B$. Therefore, there exists a function $f \in \mathcal{D}_s$ with support in B such that $g = \bar{R}f$. The injectivity of the operator $\bar{R}$ shows that the same f works for each such ball. Therefore, there exists an $f \in \mathcal{D}_s$ such that $g = \bar{R}f$ and f is supported within every closed ball containing K . Since any compact convex set K is the intersection of all the closed balls containing it, f is supported in K and the theorem is established. ■

4.2.1 Application to the Paley–Wiener Theorem

The Paley–Wiener theorems give consistency conditions for the range of the Fourier transform in the same way that theorem 4.12 gives consistency conditions for the range of the Radon transform.

This section shows how to derive an n -dimensional version of the Paley–Wiener theorem based on the one-dimensional version and the range theorems for the Radon transform.

The Paley–Wiener theorems concern functions of exponential type and the Fourier transform with support on symmetric, convex compact sets.

Definition 4.13. A subset K of $\mathbb{R}^n$ is said to be symmetric if $-x \in K$, whenever $x \in K$.

That the only symmetric, compact, convex subsets of the real line are intervals of the form $[-k, k]$ is easily checked.

Definition 4.14. Let k be a positive real number. A function $F : \mathbb{C} \rightarrow \mathbb{C}$ of one complex variable is said to be of *exponential type k* if it is an entire analytic function and $\forall \varepsilon > 0, \exists A_\varepsilon > 0$ such that $\forall z \in \mathbb{C}$,

$$|F(z)| \leq A_\varepsilon e^{(k+\varepsilon)|z|} \quad (4.16)$$

There are some equivalent formulations of this definition. Before stating these, let us extend the definition of the support function to complex values. We do this only for symmetric, compact, convex sets. If K is such a set, then we have already defined $\|\xi\|_K = \sup_{x \in K} \langle \xi, x \rangle$ for $\xi \in \mathbb{R}^n$. If K is symmetric, then this definition is equivalent to $\|\xi\|_K = \sup_{x \in K} |\langle \xi, x \rangle|$, since if $\langle \xi, x \rangle < 0$, then $|\langle \xi, x \rangle| = \langle \xi, -x \rangle > 0$ and by the symmetry $-x \in K$ also. Therefore, we extend the definition of the support function to

complex valued vectors and compact, convex sets satisfying the symmetry condition by defining

$$\|z\|_K = \sup_{x \in K} |\langle z, x \rangle|$$

for $z \in \mathbb{C}^n$.

Proposition 4.15. *Let $k > 0$ and let $K = [-k, k]$. An entire analytic function $F : \mathbb{C} \rightarrow \mathbb{C}$ is of exponential type k if and only if*

$$\limsup_{|z| \rightarrow \infty} \frac{\ln |F(z)|}{|z|} \leq k. \quad (4.17)$$

and if and only if

$$\limsup_{|z| \rightarrow \infty} \frac{\ln |F(z)|}{\|z\|_K} \leq 1. \quad (4.18)$$

Proof. If condition (4.17) is satisfied, then for any $\varepsilon > 0$, there is an R_0 such that $|z| \geq R_0$ implies $\sup_{|z| \geq R_0} \frac{\ln |F(z)|}{|z|} < k + \varepsilon$ and this implies that $\ln |F(z)| < |z|(k + \varepsilon)$ for $|z| \geq R_0$. Take the exponential of both sides to get $|F(z)| \leq e^{(k+\varepsilon)|z|}$ for $|z| \geq R_0$. Because the set where $|z| \leq R_0$ is compact and $\frac{|F(z)|}{e^{(k+\varepsilon)|z|}}$ is continuous, we see that there is a constant $A_\varepsilon \geq 1$ that bounds this quotient on the disk of radius R_0 . Thus on this disk we have $|F(z)| \leq A_\varepsilon e^{(k+\varepsilon)|z|}$, whereas on the complement we have $|F(z)| \leq e^{(k+\varepsilon)|z|} \leq A_\varepsilon e^{(k+\varepsilon)|z|}$ since $A_\varepsilon \geq 1$. This proves that F is of exponential type k . The proof of the converse is of a similar nature.

If we observe that the support function of the interval $K = [-k, k] \subset \mathbb{R}$ is $\|\xi\|_K = k|\xi|$, then the equivalency of the last condition follows immediately. ■

Based on proposition 4.15, particularly equation (4.18), we generalize the definition of functions of exponential type to dimension n .

Definition 4.16. Let K be a symmetric, compact, convex subset of $\mathbb{R}^n$. A function $F : \mathbb{C}^n \rightarrow \mathbb{C}$ of one complex variable is said to be of **exponential type** K if it is an entire analytic function such that

$$\limsup_{|z| \rightarrow \infty} \frac{\ln |F(z)|}{\|z\|_K} \leq 1. \quad (4.19)$$

We note that by definition 4.14 and proposition 4.15 a function F of a single complex variable is of exponential type $k > 0$, if and only if F is of exponential type $K = [-k, k]$.

Here is a statement of the Paley–Wiener theorem in dimension $n = 1$. We refer the reader to Stein and Weiss [583] or Dym and McKean [138] for the proof.

Theorem 4.17 (Paley–Wiener, $n = 1$). *Let F be an L^2 function on $\mathbb{R}$. Then F is the Fourier transform of an L^2 function supported on the interval $K = [-k, k]$, if and only if F extends to a function of exponential type K on $\mathbb{C}$.*

The Paley–Wiener theorem in dimension $n = 1$ generalizes to any dimension by replacing the interval K by a symmetric, compact, convex set. Stein and Weiss [583]

have a proof of this extension that is based on the theory of H^2 functions. However, we prefer to present the proof of the following version of the Paley–Wiener theorem, which is based on our theory of the range of the Radon transform. This proof is from Smith, Solmon, and Wagner [566].

Theorem 4.18 (Paley–Wiener, dimension n). *Let K be a symmetric, compact, convex subset of $\mathbb{R}^n$. Then F is the Fourier transform of an L^2 function f supported on K , if and only if F satisfies the following three conditions:*

1. $F \in C^\infty \cap L^2$ on $\mathbb{R}^n$,
2. for almost every direction $\theta \in S^{n-1}$ the function $F_\theta(\tau) = F(\tau\theta)$ extends to an entire analytic function on $\mathbb{C}$, and
3. for almost every direction $\theta \in S^{n-1}$

$$\limsup_{\substack{|\tau| \rightarrow \infty \\ \tau \in \mathbb{C}}} \frac{\ln |F(\tau\theta)|}{|\tau|} \leq \|\theta\|_K.$$

Before proving theorem 4.18, let us establish the following result.

Lemma 4.19. *Let F be a C^∞ function on $\mathbb{R}^n$ and let $F_\theta(\tau) = F(\tau\theta)$. If F_θ extends to an entire analytic function on $\mathbb{C}$, then*

$$F_\theta(\tau) = \sum_{k=0}^{\infty} \langle \theta, \partial \rangle^k F(0) \frac{\tau^k}{k!} \quad (4.20)$$

If we assume further that $F = \widehat{f}$ for a compactly supported L^1 function f , then

$$F_\theta(\tau) = (2\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \int_{\mathbb{R}^n} \langle \theta, -ix \rangle^k f(x) dx. \quad (4.21)$$

Proof. By hypothesis we can expand $F_\theta(\tau)$ in a Maclaurin series:

$$F_\theta(\tau) = \sum_{k=0}^{\infty} F_\theta^{(k)}(0) \frac{\tau^k}{k!}$$

But $F_\theta(\tau) = F(\tau\theta)$, so, by the chain rule, $F'_\theta(\tau) = \sum_{j=1}^n \theta_j \frac{\partial F}{\partial x_j}(\tau\theta)$. This can be written as $F'_\theta(\tau) = \langle \theta, \partial \rangle F(\tau\theta)$. In the same way, we can calculate the second derivative: $F''_\theta(\tau) = \langle \theta, \partial \rangle (\langle \theta, \partial \rangle F)(\tau\theta) = \langle \theta, \partial \rangle^2 F(\tau\theta)$. By induction we see that, in general, $F_\theta^{(k)}(\tau) = \langle \theta, \partial \rangle^k F(\tau\theta)$ and, therefore, $F_\theta(\tau) = \sum_{k=0}^{\infty} \langle \theta, \partial \rangle^k F(0) \frac{\tau^k}{k!}$, thus establishing equation (4.20).

Assuming that $F = \widehat{f}$, we can use the derivative theorem for the Fourier transform with $p(x) = \langle \theta, x \rangle^k$ to get

$$\langle \theta, \partial \rangle^k F(0) = p(\partial) \widehat{f}(0) = (p(-ix) f(x))^\wedge(0)$$

If we have the extra condition that f is a compactly supported L^1 function, then we can write the last expression in integral form as $(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \langle \theta, -ix \rangle^k f(x) dx$. Putting this back into the equation establishes equation (4.21). ■

Proof of Paley–Wiener, Theorem 4.18. We first prove the sufficiency of conditions 1–3.

Consider the function $|\xi|^{-\frac{n-1}{2}} F(\xi)$. Near the origin $F(\xi)$ is bounded, being C^∞ , and $|\xi|^{-\frac{n-1}{2}}$ is L^2 . Therefore, the product is L^2 near the origin. Away from the origin, the roles of the functions are reversed, and again the product is L^2 . Therefore, from polar coordinates we have $\infty > \int_{\mathbb{R}^n} ||\xi|^{-\frac{n-1}{2}} F(\xi)|^2 d\xi = \int_{S^{n-1}} \int_0^\infty |F(\tau\theta)|^2 d\tau d\theta$. Consequently, by Fubini's theorem, for almost every θ the function $F_\theta(\tau) = F(\tau\theta)$ is L^2 on $\mathbb{R}$. Hence, for each such θ , there is a function $g_\theta \in L^2(\mathbb{R})$ such that $\widehat{g_\theta}(\tau) = F_\theta(\tau) = F(\tau\theta)$.

Define $g(\theta, t) = g_\theta(t)$ on $S^{n-1} \times \mathbb{R}$. Again, using polar coordinates, we have

$$\begin{aligned} \left\| |\tau|^{\frac{n-1}{2}} \widehat{g}(\theta, \tau) \right\|_{L^2(S^{n-1} \times \mathbb{R})}^2 &= \int_{S^{n-1}} \int_{-\infty}^{\infty} |F(\tau\theta)|^2 |\tau|^{n-1} d\tau d\theta \\ &= 2 \|F\|_{L^2(\mathbb{R}^n)}^2 < \infty. \end{aligned}$$

A similar calculation shows that $g \in L^2$. This shows that $g \in \mathcal{D}_{\frac{n-1}{2}}$. An easy calculation shows that g is even. Thus g satisfies the first consistency condition of theorem 4.12.

The condition $\limsup \frac{\ln |F_\theta(\tau)|}{|\tau|} \leq \|\theta\|_K$ shows that g_θ satisfies the condition of the one-dimensional Paley–Wiener theorem on the interval $[-\|\theta\|_K, \|\theta\|_K]$ and, therefore, $g(\theta, t) = g_\theta(t) = 0$ for $|t| > \|\theta\|_K$. Thus, g satisfies the second consistency condition of theorem 4.12.

For the third consistency condition we attempt to expand $\widehat{g_\theta}(\tau)$ in a power series. We note that $g_\theta(t)$ is an L^2 function supported on the compact set $[-\|\theta\|_K, \|\theta\|_K]$. Therefore, $g_\theta(t)$ is an L^1 function and, hence, $\|g_\theta\|_{L^1}$ is finite. Furthermore, we can compute the Fourier transform in integral form: $\widehat{g_\theta}(\tau) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} g_\theta(t) e^{-it\tau} dt$. We next expand $e^{-it\tau}$ in its Maclaurin series and integrate the result term by term. We will justify the term-by-term integration presently, but the result is:

$$\widehat{g_\theta}(\tau) = \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} t^k g_\theta(t) dt \right) (2\pi)^{-\frac{1}{2}} (-i)^k \frac{\tau^k}{k!} \quad (4.22)$$

We now justify the term by term integration. This can be done by showing that the sum of the integrals of the absolute series for $g_\theta(t)e^{-it\tau}$ converges. But that series is precisely $\sum_{k=0}^{\infty} \int_{-\infty}^{\infty} |g_\theta(t)| \frac{|t\tau|^k}{k!} dt$. In the following argument, a represents the maximum of $\|\theta\|_K$ and 1. Therefore, g_θ is an L^1 function supported in $[-a, a]$ and we have:

$$\begin{aligned} \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} |g_\theta(t)| \frac{|t\tau|^k}{k!} dt &= \sum_{k=0}^{\infty} \int_{-a}^a |g_\theta(t)| \frac{|t\tau|^k}{k!} dt \\ &\leq \sum_{k=0}^{\infty} \frac{|\tau a|^k}{k!} \int_{-a}^a |g_\theta(t)| dt = e^{|\tau|a} \|g_\theta\|_{L^1} < \infty. \end{aligned}$$

This justifies the calculation resulting in equation (4.22).

Recall that $\widehat{g}_\theta(\tau)$ also equals $F_\theta(\tau)$, which, by hypothesis, extends to an entire analytic function on $\mathbb{C}$. Therefore, there is enough information to use equation (4.20) of lemma 4.19 to get

$$\widehat{g}_\theta(\tau) = \sum_{k=0}^{\infty} \langle \theta, \partial \rangle^k F(0) \frac{\tau^k}{k!} \quad (4.23)$$

If we now compare coefficients of the series (4.22) and (4.23), we see that $\int_{-\infty}^{\infty} t^k g_\theta(t) dt = (2\pi)^{\frac{1}{2}} i^k \langle \theta, \partial \rangle^k F(0)$, but it is easy to see from the binomial formula that $\langle \theta, \partial \rangle^k F(0)$ is a homogeneous polynomial in θ of degree k . Thus g satisfies the last consistency condition of theorem 4.12. Hence, an L^2 function h exists with support on K , such that $\overline{R}h = g$.

Let f be the inverse Fourier transform of F . Since F is L^2 , then f exists and is L^2 . We can now show that f has support on K . By the slice-projection theorem we get

$$\begin{aligned} \widehat{h}(\tau\theta) &= (2\pi)^{-\frac{n-1}{2}} \widehat{\overline{R}h}(\tau, \theta) = (2\pi)^{-\frac{n-1}{2}} \widehat{g}(\tau, \theta) \\ &= (2\pi)^{-\frac{n-1}{2}} F(\tau\theta) = (2\pi)^{-\frac{n-1}{2}} \widehat{f}(\tau\theta). \end{aligned}$$

Thus, the Fourier transforms of f and $(2\pi)^{\frac{n-1}{2}} h$ agree almost everywhere. By taking inverse Fourier transforms we see that f and $(2\pi)^{\frac{n-1}{2}} h$ agree almost everywhere. Because h is supported on K , the same is true of f and we have proved the sufficiency of conditions 1–3.

We now show that conditions 1–3 of theorem 4.18 are necessary.

Suppose, therefore, that F is the Fourier transform of an L^2 function f supported on the symmetric, compact, convex set $K \subset \mathbb{R}^n$. Because f is a compactly supported L^2 function it is also L^1 . We can therefore use lemma 4.5, which shows that conditions 1 and 2 are satisfied. We also have the necessary hypotheses to use equation (4.21) of lemma 4.19 to get $F_\theta(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \int_{\mathbb{R}^n} \langle \theta, -ix \rangle^k f(x) dx$. If we recall that $\|z\|_K = \sup_{x \in K} |\langle z, x \rangle|$, then $|\langle \theta, -ix \rangle| \leq \|\theta\|_K$ and we thereby have

$$\begin{aligned} |F_\theta(\tau)| &\leq c \sum_{k=0}^{\infty} \frac{|\tau|^k}{k!} \int_{\mathbb{R}^n} |\langle \theta, -ix \rangle|^k |f(x)| dx \\ &\leq c \sum_{k=0}^{\infty} \frac{|\tau| \|\theta\|_K^k}{k!} \int_{\mathbb{R}^n} |f(x)| dx \\ &= c \|f\|_{L^1} e^{|\tau| \|\theta\|_K} \end{aligned}$$

This shows that F_θ is of exponential type $\|\theta\|_K$ and from proposition 4.15 we get

$$\limsup_{|\tau| \rightarrow \infty} \frac{\ln |F_\theta(\tau)|}{|\tau|} \leq \|\theta\|_K.$$

■

As an immediate corollary we get the following more traditional statement of the n -dimensional Paley–Wiener theorem.

Theorem 4.20 (Paley–Wiener). *Let K be a symmetric, compact, convex subset of $\mathbb{R}^n$ and let F be an L^2 function on $\mathbb{R}^n$. Then F is the Fourier transform of an L^2 function f supported on K if and only if F extends to a function of exponential type K on $\mathbb{C}^n$.*

The paper of Arguedas and Estrada [25] is somewhat related to the ideas in this section. They prove a support theorem for Radon transforms on $\mathbb{R}^2$ and show that if $f(x, y)$ is a real analytic function on the plane $\mathbb{R}^2$ for which the restriction f_θ of f to the line through the origin in direction θ extends to an entire function on the complexified line, for a small set of directions θ , then f itself extends to an entire function on $\mathbb{C}^2$.

Remark 4.21. Although the conditions in theorem 4.18 appear weaker than those in theorem 4.20, they are in fact equivalent, because both conditions are equivalent to f having support on K and having a square integrable Fourier transform. However, the weaker statement is sometimes useful in applications. For example, we will use theorem 4.18 in a lemma that helps prove theorem 4.24 on the range characterization of the k -plane transform.

4.2.2 The Helgason–Ludwig–Solmon Range Theorem for the k -Plane Transform

Helgason [267, 269] derived consistency conditions for a rapidly decreasing function g to be in the range of the Radon transform. These conditions are of the form: $p_m(\theta) = \int_{-\infty}^{\infty} t^m g(\theta, t) dt$ is the restriction of a homogeneous polynomial of degree m to the unit sphere, $m \in \mathbb{Z}^+$. Helgason also showed that these conditions and evenness of f are also sufficient for g to be in the range of the Radon transform. Ludwig [405] showed how to characterize the range of the Radon transform on C^∞ functions supported on a convex, compact set. Solmon [569] proved the same theorem for the k -plane transform with $k < n - 1$. We now present a single theorem containing both cases.

The statement of the Helgason–Ludwig–Solmon range theorem (theorem 4.24) will seem to be very similar to that of theorem 4.3. However, theorem 4.3 only gives sufficient conditions to have $g = \overline{P}f$ and there is no guarantee that the resulting f would be compactly supported.

The next lemma, which may be of independent interest, is crucial for proving the Helgason–Ludwig–Solmon range theorem.

Lemma 4.22. *Let $\overline{P}$ be the extended k -plane transform, $0 < k < n - 1$, let K be a compact, convex subset of $\mathbb{R}^n$, and let $f \in \mathcal{D}_{-\frac{k}{2}}$ be such that*

1. $\overline{P}f(\eta, x'') = 0$ whenever $x'' + \eta$ does not meet K , and

2. for each nonnegative integer m , a homogeneous polynomial Q_m of degree m exists on $\mathbb{R}^n$ such that the restriction of Q_m to any $n - k$ plane $\eta^\perp$ satisfies

$$Q_m(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m \overline{P}f(\eta, x'') dx''$$

for $\xi'' \in \eta^\perp$.

Then f has support in K .

Proof. We first deal with the special case in which K is symmetric. For this we plan to use the Paley-Wiener theorem 4.18 in the following way.² We construct a function F which satisfies conditions 1, 2, and 3 of theorem 4.18 and such that $\widehat{f}(\xi) = F(\xi)$ when $\xi \in \mathbb{R}^n$. It will follow that f has support in K .

Define $F(\tau) = (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\tau)$ for $\tau \in \mathbb{C}^n$. Condition 1 of this lemma implies that for fixed η , the function $x'' \mapsto \overline{P}f(\eta, x'')$ is compactly supported on $\eta^\perp$. Therefore, $\overline{P}f(\eta, \cdot)$ is in $L^1(\eta^\perp)$ and we can then use remark 4.6 to apply lemma 4.5 to the $(n - k)$ -plane $\eta^\perp$ to show that for $\xi'' \in \eta^\perp$ we have $\widehat{\overline{P}f}(\eta, \xi'') = (2\pi)^{-\frac{n-k}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} \int_{\eta^\perp} \langle \xi'', x'' \rangle^m \overline{P}f(\eta, x'') dx''$. Furthermore, lemma 4.5 shows that the infinite series appearing in the last equation is convergent for all $\xi'' \in \eta^\perp$. By condition 2 of this lemma we can replace $\int_{\eta^\perp} \langle \xi'', x'' \rangle^m \overline{P}f(\eta, x'') dx''$ by $Q_m(\xi'')$ and as a consequence $\widehat{\overline{P}f}(\eta, \xi'') = (2\pi)^{-\frac{n-k}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\xi'')$. This shows that $F(\xi'')$ is defined and finite for $\xi'' \in \eta^\perp$. Every ξ in $\mathbb{R}^n$ is in some $(n - k)$ -plane $\eta^\perp$ and by condition 2 of this lemma each Q_m is a homogeneous polynomial defined on $\mathbb{R}^n$. As a consequence, F is well defined, independent of the choice of $\eta^\perp$, and has a convergent power series expansion at every point of $\mathbb{R}^n$. Hence F extends to an entire analytic function on $\mathbb{C}^n$ with the same power series expansion. In particular F restricted to $\mathbb{R}^n$ is C^∞ which validates the first part of condition 1 of the Paley-Wiener theorem. Also we immediately obtain the fact that $F_\theta(\xi) = F(\xi\theta)$ extends to an entire analytic function on $\mathbb{C}$ for every direction $\theta \in S^{n-1}$, which implies condition 2 of the Paley-Wiener theorem.

Because $f \in \mathcal{D}_{-\frac{k}{2}}$, we can use the extended slice-projection theorem to show that $\widehat{\overline{P}f}(\eta, \xi'') = (2\pi)^{\frac{k}{2}} \widehat{f}(\xi'')$ and, hence, we obtain the equation

$$\widehat{f}(\xi'') = (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{i^{-m}}{m!} Q_m(\xi'') = F(\xi'')$$

for all $\xi'' \in \eta^\perp$. As before this is true for all $\xi \in \mathbb{R}^n$ and we therefore have

$$\widehat{f}(\xi) = F(\xi) \tag{4.24}$$

for all $\xi \in \mathbb{R}^n$. Again because $f \in \mathcal{D}_{-\frac{k}{2}}$, then $f \in L^2$ which implies that $F \in L^2$ and this completes the validation that F satisfies condition 1 of the Paley-Wiener theorem.

² Our proof of the Paley-Wiener theorem 4.18 required the range theorem for the Radon transform. Therefore, to avoid circularity we need to assume that $k < n - 1$. This is the only point where we need to make this assumption, compare remark 4.25.

To verify condition 3 of the Paley–Wiener theorem we compute a growth estimate for $|F_\theta(\tau)|$ where τ is a complex number. Complexify $\eta^\perp$ by defining

$$\eta_{\mathbb{C}}^\perp = \left\{ \zeta'' = \xi'' + i\omega'' : \xi'', \omega'' \in \eta^\perp \right\}.$$

Then we can regard $\eta_{\mathbb{C}}^\perp$ as a copy of $\mathbb{C}^{n-k}$ in the same way that we regard $\eta^\perp$ as a copy of $\mathbb{R}^{n-k}$. The homogeneous polynomial $Q_m(\xi'')$ extends from $\eta^\perp$ to a homogeneous polynomial, and, hence, an analytic function on $\eta_{\mathbb{C}}^\perp$. If we define $S_m(\zeta'')$ on $\eta_{\mathbb{C}}^\perp$ by $S_m(\zeta'') = \int_{\eta^\perp} \langle \zeta'', x'' \rangle^m \overline{P}f(\eta, x'') dx''$ then $S_m(\zeta'')$ is an analytic function on $\eta_{\mathbb{C}}^\perp$. To see this we apply the Cauchy–Riemann operator $\bar{\partial}$. It can be taken inside the integral because $\overline{P}f(\eta, x'')$ has compact support and is integrable on $\eta^\perp$. By the Cauchy–Riemann equations, $\bar{\partial}(\langle \zeta'', x'' \rangle^m) = 0$, so $\bar{\partial}S_m = 0$ and, hence, S_m is analytic. We now see that both $S_m(\zeta'')$ and $Q_m(\zeta'')$ are entire analytic functions on $\eta_{\mathbb{C}}^\perp$, which agree on $\eta^\perp$. By the identity theorem for analytic functions they are equal and, hence,

$$Q_m(\zeta'') = \int_{\eta^\perp} \langle \zeta'', x'' \rangle^m \overline{P}f(\eta, x'') dx'' \text{ for } \zeta'' \in \eta_{\mathbb{C}}^\perp.$$

So if $\zeta'' \in \eta_{\mathbb{C}}^\perp$, then

$$|Q_m(\zeta'')| \leq \int_{\eta^\perp} |\langle \zeta'', x'' \rangle|^m |\overline{P}f(\eta, x'')| dx'' \leq c_1 \|\zeta''\|_K^m$$

where $c_1 = \int_{\eta^\perp} |\overline{P}f(\eta, x'')| dx''$. The constant is finite because condition 1 implies that $\overline{P}f(\eta, x'')$ is an L^1 function on $\eta^\perp$. Therefore, for all $\zeta'' \in \eta_{\mathbb{C}}^\perp$,

$$\begin{aligned} |F(\zeta'')| &\leq c_1 (2\pi)^{-\frac{n}{2}} \sum_{m=0}^{\infty} \frac{1}{m!} \|\zeta''\|_K^m \\ &= c \exp(\|\zeta''\|_K). \end{aligned} \tag{4.25}$$

where $c = c_1 (2\pi)^{-\frac{n}{2}}$.

Now let us be given an arbitrary direction $\theta \in S^{n-1}$. We can find some $\eta \in G_{k,n}$ such that $\tau\theta \in \eta_{\mathbb{C}}^\perp$ for all complex numbers τ . This is in fact very simple: we can find an $n - k$ dimensional space v containing θ . Let η be the orthogonal complement of v . Then $\eta \in G_{k,n}$ and both $(\operatorname{Re} \tau)\theta$ and $(\operatorname{Im} \tau)\theta \in \eta^\perp$. Hence $\tau\theta \in \eta_{\mathbb{C}}^\perp$.

Having done this, then using (4.25) and letting $a = \ln c$, we have $|F_\theta(\tau)| \leq \exp(|\tau|\|\theta\|_K + a) = \exp(|\tau|(\|\theta\|_K + \frac{a}{|\tau|}))$. Taking logarithms we get $\frac{\ln |F_\theta(\tau)|}{|\tau|} \leq \|\theta\|_K + \frac{a}{|\tau|}$ which clearly implies $\limsup_{\tau \in \mathbb{C}} \frac{\ln |F_\theta(\tau)|}{|\tau|} \leq \|\theta\|_K$. This verifies the last condition of the Paley–Wiener theorem. Since K is a symmetric, compact, convex subset of $\mathbb{R}^n$ we can conclude from the Paley–Wiener theorem 4.18 that F restricted to $\mathbb{R}^n$ is the Fourier transform of an L^2 function supported on K . Therefore, the inverse Fourier transform of F , which by equation (4.24) we know is f , must be supported on K .

This completes the proof of the lemma in the case that K is a symmetric, compact, convex set. The next step is to prove the result where K is any closed ball in $\mathbb{R}^n$. Because a closed ball centered at the origin is a symmetric, compact, convex set, the plan is to use the lemma for that case to prove that the lemma is true for any translate of such a ball.

So let a be any point in $\mathbb{R}^n$, let B be a closed ball with center at a and let f be a function that satisfies conditions 1 and 2 relative to $K = B$. We will show that f has support in B .

Let B^* be the closed ball that arises by translating B to have center at the origin. Define $f^*(x) = f(x + a)$. It is easy to check that $f^* \in \mathcal{D}_{-\frac{k}{2}}$. Also the translation theorem for the k -plane transform shows that f^* satisfies condition 1.

To verify condition 2 we define $Q_m^*(\xi) = \sum_{j=0}^m \binom{m}{j} \langle \xi, -a \rangle^{m-j} Q_j(\xi)$ on $\mathbb{R}^n$. Because the polynomials Q_j are homogeneous of degree j and $\langle \xi, -a \rangle^{m-j}$ is a homogeneous polynomial of degree $m-j$ in ξ , then Q_m^* is a homogeneous polynomial of degree m . Let $\xi'' \in \eta^\perp$, and let $a'' = \text{proj}_{\eta^\perp} a$. Then $\langle \xi'', -a \rangle = \langle \xi'', -a'' \rangle$ and hence

$$Q_m^*(\xi'') = \sum_{j=0}^m \binom{m}{j} \langle \xi'', -a'' \rangle^{m-j} Q_j(\xi'')$$

Using condition 2 for the function f then gives

$$\begin{aligned} Q_m^*(\xi'') &= \sum_{j=0}^m \binom{m}{j} \langle \xi'', -a'' \rangle^{m-j} \int_{\eta^\perp} \langle \xi'', x'' \rangle^j \overline{P} f(\eta, x'') dx'' \\ &= \sum_{j=0}^m \binom{m}{j} \langle \xi'', -a'' \rangle^{m-j} \int_{\eta^\perp} \langle \xi'', x'' \rangle^j \overline{P} f^*(\eta, x'' - a'') dx''. \end{aligned}$$

The last step used the translation theorem for the k -plane transform to replace $\overline{P} f(\eta, x'')$ by $\overline{P} f^*(\eta, x'' - a'')$. In the next step we substitute x'' for $x'' - a''$ and use the binomial theorem to get

$$\begin{aligned} Q_m^*(\xi'') &= \int_{\eta^\perp} \sum_{j=0}^m \binom{m}{j} \langle \xi'', -a'' \rangle^{m-j} \langle \xi'', x'' + a'' \rangle^j \overline{P} f^*(\eta, x'') dx'' \\ &= \int_{\eta^\perp} (\langle \xi'', -a'' \rangle + \langle \xi'', x'' + a'' \rangle)^m \overline{P} f^*(\eta, x'') dx'' \\ &= \int_{\eta^\perp} \langle \xi'', x'' \rangle^m \overline{P} f^*(\eta, x'') dx''. \end{aligned}$$

This proves that condition 2 of this lemma is valid for the function f^* .

Since B^* is a symmetric, compact, convex set and since the conditions of the lemma have been verified for f^* , then f^* has support in B^* . This immediately implies that f has support in B and completes the proof that we can replace the set K in the lemma by any closed ball in $\mathbb{R}^n$.

Finally, let K be any compact, convex subset of $\mathbb{R}^n$ and let $f \in \mathcal{D}_{-\frac{k}{2}}$ be such that conditions 1 and 2 are satisfied for f relative to K . Let B be any closed ball containing K . Then condition 1 is obviously satisfied for f relative to B . The other condition is independent of both K and B and, hence, is satisfied. Therefore, we can conclude that f has support in B . Because K is the intersection of all closed balls containing K , it then follows that f has support in K . ■

Corollary 4.23. *If f satisfies the conditions of the lemma, then $\overline{P}f = Pf$.*

Proof. Since the lemma implies that f is a compactly supported L^2 function, then the extended k -plane transform can be computed by the integral defining the usual k -plane transform because of corollary 3.96. ■

Theorem 4.24 (Helgason–Ludwig–Solmon range theorem). *Let $\overline{P}$ be the extended k -plane transform, $0 < k \leq n - 1$, let K be a compact, convex subset of $\mathbb{R}^n$, and let g be a measurable function on $\mathcal{G}_{k,n}$. Then $g = \overline{P}f$ for some $f \in \mathcal{D}_s$ with support in K if and only if*

1. $g \in \mathcal{D}_{\frac{k}{2}+s}$,
2. $g(\eta, x'') = 0$ whenever $x'' + \eta$ does not meet K , and
3. for each nonnegative integer m , a homogeneous polynomial Q_m of degree m exists on $\mathbb{R}^n$ such that the restriction of Q_m to any $n - k$ -plane $\eta^\perp$ satisfies

$$Q_m(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''$$

for $\xi'' \in \eta^\perp$.

Proof. The conditions are necessary: Let $g = \overline{P}f$ for some $f \in \mathcal{D}_s$ with support in K . Since this implies that $\overline{P}f$ exists, then $f \in \mathcal{D}_{-\frac{k}{2}}$. We can therefore use the isometry theorem (theorem 3.97) with $s = -\frac{k}{2}$ to show that $g \in L^2(\mathcal{G}_{k,n})$. This and the isometry theorem (theorem 3.97) with the current value of s show that $g \in \mathcal{D}_{\frac{k}{2}+s}$, thereby proving condition 1. The support condition on f obviously implies condition 2. Finally, because $f \in L^2$ and the support K of f is compact, then $f \in L^1_0$ and we can use lemma 4.5 to show that condition 3 is true.

We now prove the sufficiency of the conditions.

If $k = n - 1$, then we essentially have the Radon transform and theorem 4.12 proves the sufficiency of the consistency conditions.

If $k < n - 1$ and g satisfies conditions 1–3, then we can use theorem 4.3 to find some $f \in \mathcal{D}_s \cap \mathcal{D}_{-\frac{k}{2}}$ with $g = Pf$. Then conditions 2 and 3 of this theorem imply that hypotheses 1 and 2 of lemma 4.22 are valid. This, together with the fact that $f \in \mathcal{D}_{-\frac{k}{2}}$, shows that we can use lemma 4.22 to conclude that the support of f is contained in K . ■

Remark 4.25. It was necessary for us to prove the case $k = n - 1$ (theorem 4.12) separately from the case $k < n$, because we needed the Smith–Solmon–Wagner version

of the Paley–Wiener theorem to derive lemma 4.22 and this Paley–Wiener theorem depended on the sufficiency of the consistency conditions derived in theorem 4.12 for $k = n - 1$.

Solmon [569], taking a slightly different approach, used the classical n -dimensional Paley–Wiener theorem, which can be proved independently of the theory of the Radon transform.³ However, to use the Paley–Wiener theorem, Solmon required any $\xi \in \mathbb{C}^n$ to be in at least one of the complexified k -planes $\eta_{\mathbb{C}}^{\perp}$. It is easy to see that this is possible if and only if $k < n - 1$.

If one could prove Smith’s version of the Paley–Wiener theorem 4.18 without recourse to a range result for the Radon transform, then our proof would work for any k -plane transform, including the Radon transform.

The following functional analytic interpretation of this theorem is of interest.

Definition 4.26. $\mathcal{D}_s^H(\mathcal{G}_{k,n})$ consists of the set of all measurable functions g defined on $\mathcal{G}_{k,n}$ such that

- (1) $g \in \mathcal{D}_s(\mathcal{G}_{k,n})$,
- (2) A compact, convex set exists in $\mathbb{R}^n$ such that $g(\eta, x'') = 0$ whenever $x'' + \eta$ does not meet K , and
- (3) for each nonnegative integer m , a homogeneous polynomial Q_m of degree m exists on $\mathbb{R}^n$ such that the restriction of Q_m to any $n - k$ -plane $\eta^{\perp}$ satisfies

$$Q_m(\xi'') = \int_{\eta^{\perp}} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''.$$

As usual the subscript 0 indicates compactly supported functions, so $\mathcal{D}_{s,0}(\mathbb{R}^n)$ denotes all compactly supported functions in $\mathcal{D}_s(\mathbb{R}^n)$.

Theorem 4.27. *The k -plane transform P is a bijection of $\mathcal{D}_{s,0}(\mathbb{R}^n)$ onto $\mathcal{D}_{\frac{k}{2}+s}^H(\mathcal{G}_{k,n})$.*

Proof. First note that P maps $\mathcal{D}_{s,0}(\mathbb{R}^n)$ into $\mathcal{D}_{\frac{k}{2}+s}^H(\mathcal{G}_{k,n})$. For if $f \in \mathcal{D}_{s,0}(\mathbb{R}^n)$ and if K denotes the convex hull of the (compact) support of f , then the work we did in the proof of theorem 4.24 shows that conditions 1 and 3 of definition 4.26 are satisfied for $g = Pf$. Also condition 2 is satisfied because $Pf(\eta, x'') = 0$ whenever $x'' + \eta$ does not meet the support of f and K contains this support set. Therefore, $P(\mathcal{D}_{s,0}(\mathbb{R}^n)) \subset \mathcal{D}_{\frac{k}{2}+s}^H(\mathcal{G}_{k,n})$. Theorem 4.24 also proves the surjectivity. The injectivity is a consequence of theorem 3.108. ■

We do not claim, and it is not true, that the inverse map in this theorem is continuous (see theorem 4.35).

³ Stein and Weiss [583] prove an n -dimensional version of the Paley–Wiener theorem.

4.3 Differentiability Properties of the Radon Transform and the Relation of the Radon Transform to Partial Differential Equations

We have already observed that the Radon transform is a smoothing operator: we were able to interpret corollary 3.99 to mean that the k -plane transform $Pf(\eta, x'')$ of a compactly supported function f with m derivatives has approximately $m + \frac{k}{2}$ derivatives in the x'' variable. Thus, the k -plane transform of a function is generally smoother than the function. In this section we consider two main themes:

1. What are the range properties of the Radon transform restricted to very smooth functions, say rapidly decreasing functions?
2. How may the Radon transform be used in solving partial differential equations?

Surprisingly, an intimate relation exists between these two seemingly different topics.

Unfortunately, we only have room for a survey of these ideas. Except for a few proofs that we provide, the reader is directed to the references for any omitted proofs.

We first investigate the differentiability properties of the Radon transform in some more detail.

The Laplacian operator Δ operating on $C^2(\mathbb{R}^n)$ was defined in section 3.6.1. We define the operator $\square$ on $C^2(S^{n-1} \times \mathbb{R})$ by $\square g(\theta, p) = \frac{\partial^2}{\partial p^2} g(\theta, p)$.

Theorem 4.28. *If f is a compactly supported C^∞ function, or more generally, if f is a rapidly decreasing function, then the Radon transform Rf of f is of the same nature on $S^{n-1} \times \mathbb{R}$.*

Furthermore, the operators Δ and $\square$ intertwine the Radon transform in the sense that

$$R(\Delta f) = \square(Rf). \quad (4.26)$$

Proof. For the purposes of this proof, let $\partial_\theta^\alpha g$ be the partial derivative of $g \in C^k(S^{n-1} \times \mathbb{R})$ with respect to the variable $\theta \in S^{n-1}$ and let $\partial_p^\beta g$ be the partial derivative with respect to the variable $p \in \mathbb{R}$. Since $f \in \mathcal{S}$, then corollary 3.99 implies that $\partial_p^\beta Rf(\theta, p)$ exists for all multi-indices β . The proof that $\partial_\theta^\alpha Rf(\theta, p)$ exists is fairly technical and we refer the reader to either Helgason [291], theorem 2.4, or to Carton-Lebrun [79].^{4,5} In both [291] and [79] we get not only the existence of the derivatives, but also the necessary growth estimates to verify the rapidly decreasing nature of Rf .

We use Fourier analysis to prove the intertwining property (4.26). The left and right sides of equation (4.26) are well defined and in $\mathcal{S}(S^{n-1} \times \mathbb{R})$ as a consequence of the first part of the proof. Hence, we can take the Fourier transform of both sides of this equation, as always with respect to the second variable $p \in \mathbb{R}$. Note that the Laplacian

⁴ The proof in Helgason [291] is by brute force working on coordinate patches of the sphere, whereas the proof in Carton-Lebrun [79] is coordinate free and more conceptual. Carton-Lebrun [79] gives sufficient conditions for $\int_{\mathbb{R}} \varphi(t, x) K(|x|, t) dt$ to be in $C^m(\mathbb{R}^n)$ and this leads to the required smoothness in the case of the Radon transform.

⁵ There is a misprint in [291], theorem 2.4, page 5, line 16: the expression $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$ should be $\widehat{f} \in \mathcal{S}(\mathbb{P}^n)$.

Δ is the differential operator corresponding to the polynomial $x_1^2 + \cdots + x_n^2$, so by the derivative theorem for the Fourier transform we have

$$\mathcal{F}(\Delta f)(\xi) = \sum (i\xi_j)^2 \widehat{f}(\xi) = -|\xi|^2 \widehat{f}(\xi).$$

Likewise, we get

$$\mathcal{F}(\square g)(\theta, q) = -q^2 \widehat{g}(\theta, q).$$

Hence by the slice-projection theorem

$$\begin{aligned} \mathcal{F}(R(\Delta f))(\theta, q) &= (2\pi)^{\frac{n-1}{2}} \mathcal{F}(\Delta f)(q\theta) \\ &= -(2\pi)^{\frac{n-1}{2}} q^2 \widehat{f}(q\theta) = -q^2 \widehat{Rf}(q, \theta) \\ &= \mathcal{F}(\square Rf)(\theta, q). \end{aligned}$$

Fourier inversion now proves equation (4.26). ■

4.3.1 The Range of the k -Plane Transform on Smooth Functions

We now give a brief survey of range theorems for smooth functions, directing the reader to the references for proofs.

By smooth we mean C_0^∞ or rapidly decreasing functions. When one restricts from L^2 functions to smooth functions, the range theorems become somewhat subtle. For example, theorem 4.24, in the case $s = -\frac{k}{2}$, indicates that if g is an L^2 function satisfying the conditions:

1. $g(\eta, x'') = 0$ whenever $x'' + \eta$ does not meet K ,
2. $\mathcal{Q}_m(\xi'') = \int_{\eta^\perp} \langle \xi'', x'' \rangle^m g(\eta, x'') dx''$ is a homogeneous polynomial of degree m on $\mathbb{R}^n$, $m \in \mathbb{Z}^+$

then there exists an L^2 function f such that $Pf = g$, where P is the k -plane transform. However, an example from Gonzalez [223] shows that, in the case that g is rapidly decreasing and $k < n - 1$, then these consistency conditions, by themselves, do not guarantee the existence of a rapidly decreasing function f with $Pf = g$. What is needed is a condition involving the vanishing of one or more partial differential operators acting on g . However, see the remark in section 4.5.1 about the situation in real hyperbolic spaces studied by Berenstein and Casadio Tarabusi.

Theorem 4.29 (Helgason). *The Radon transform is a bijective map from $\mathcal{S}(\mathbb{R}^n)$ to the subspace of $\mathcal{S}(S^{n-1} \times \mathbb{R})$ consisting of even functions g satisfying the consistency condition that for each $j \in \mathbb{Z}^+$, $P_j(\theta) = \int_{-\infty}^{\infty} g(\theta, s) s^j ds$ is a homogeneous polynomial of degree j on the sphere S^{n-1} .*

The Radon transform is a bijective map from $C_0^\infty(\mathbb{R}^n)$ to the subspace of $C_0^\infty(S^{n-1} \times \mathbb{R})$ consisting of even functions g satisfying the consistency condition that for each

$j \in \mathbb{Z}^+$, $P_j(\theta) = \int_{-\infty}^{\infty} g(\theta, s) s^j ds$ is a homogeneous polynomial of degree j on the sphere S^{n-1} .

This theorem is from Helgason [267, 269]. Other proofs may be found in Ludwig [405], Carton-Lebrun [79], Droste [130], and in Helgason's book [291].

Helgason has the following generalization of theorem 4.29 to the k -plane transform (see Helgason [291], theorem 6.3).

Theorem 4.30 ([291], theorem 6.3). *The k -plane transform is a bijective map from $C_0^\infty(\mathbb{R}^n)$ to the subspace of $C_0^\infty(\mathcal{G}_{k,n})$ consisting of functions g satisfying the consistency condition that for each $j \in \mathbb{Z}^+$, and $\eta \in G_{k,n}$, $P_{\eta,j}(x'') = \int_{\eta^\perp} g(\eta + u'') (x'', u'')^j du''$ is the restriction of a homogeneous polynomial of degree j on $\mathbb{R}^n$.*

When $k < n - 1$, there are equivalent descriptions of the range of the k -plane transform on $C_0^\infty(\mathbb{R}^n)$ that are characterized by the vanishing of a system of partial differential operators. The earliest result of this nature is the theorem of John (theorem 4.1), which described the range of the x-ray transform on $\mathbb{R}^4$ by such a system of partial differential equations. John's theorem was generalized to $\mathbb{R}^n$ by Kurusa [371].

The situation for the k -plane transform restricted to rapidly decreasing functions is completely different. As shown by an example of Gonzalez [224], when $k < n - 1$ there are no descriptions of the range of the k -plane transform on $\mathcal{S}(\mathbb{R}^n)$ that can be given purely in terms of moment conditions.

We will precisely state Kurusa's range theorem for the k -plane transform restricted to C_0^∞ . The statements of the range theorems for rapidly decreasing functions is more intricate and we will refer to them only in a general way. More precise statements may be found in the references.

Both John and Kurusa use a different parametrization of the k -plane transform than we used in chapter 3. They observe that a k plane is determined by almost any set of $k + 1$ points. A collection $x_0, x_1, \dots, x_k$ of $k + 1$ points in $\mathbb{R}^n$ is said to be in general position if and only if the vectors $x_j - x_0$, $j = 1, \dots, k$ are linearly independent. In this case the span of these vectors translated by the vector x_0 determines a unique k plane containing x_0 . Of course a single k plane is determined by many such sets of points. Nonetheless, we can think of the k -plane transform of f as a function of the variables $(x_0, x_1, \dots, x_k)$. We let the i th component of x_j be denoted by x_j^i . Then $\frac{\partial}{\partial x_j^i}$ denotes the corresponding partial derivative. We also let $|(x_0, x_1, \dots, x_k)|$ be the k -dimensional volume of the parallelepiped spanned by the vectors $x_j - x_0$. If the points are in general position, then this volume is nonzero.

Theorem 4.31 (Kurusa [371]). *The function $v \in C_0^\infty(\mathcal{G}_{k,n})$ is in the range of the k -plane transform P restricted to $C_0^\infty(\mathbb{R}^n)$ if and only if v satisfies the following system of partial differential equations:*

$$\left(\frac{\partial^2}{\partial x_j^i \partial x_s^i} - \frac{\partial^2}{\partial x_j^i \partial x_s^i} \right) \frac{v(x_0, x_1, \dots, x_k)}{|(x_0, x_1, \dots, x_k)|} = 0.$$

Remark 4.32. Clearly the case $k = 1$ gives John's theorem.

As another example we give a brief summary of Grinberg's range theorem [238]. Grinberg derives criteria for rapidly decreasing functions to be in the range of the k -plane transform. These conditions are generalizations of John's characterization of the range of the Radon transform in terms of ultrahyperbolic systems of partial differential equations. Grinberg parametrizes a k plane as the image of the affine transformation $y = Ax + c$, where $A = (\alpha_{ij})$ is an $(n - k) \times k$ matrix and $c \in \mathbb{R}^{n-k}$. His analogue of John's ultrahyperbolic system is $\frac{\partial^2 g}{\partial c_i \partial \alpha_{ji}} - \frac{\partial^2 g}{\partial c_j \partial \alpha_{li}} = 0$. The range of the k -plane transform then consists of solutions to this ultrahyperbolic system which satisfy certain moment conditions (compare, Grinberg [237] for analogous results for the k -plane transform on $\mathbb{C}^n$).

Gel'fand, Gindikin, and Graev [188], Grinberg [238], Richter [519], and Gonzalez [223, 224] have analogous theorems characterizing the range of the k -plane transform restricted to $\mathcal{S}(\mathbb{R}^n)$ in terms of differential operators. All but [223] describe the range in terms of large systems of second-order partial differential operators, whereas Gonzalez [223] is able to characterize the range of the k -plane transform in terms of a single fourth-order differential operator. Even more, the same operator works for all k (compare, Richter [520], Grinberg [237], and Gonzalez [222]).

Papers Gel'fand et al. [188], Grinberg [238], and Richter [519] use a less convenient parametrization of $\mathcal{G}_{k,n}$ than Kurusa [371]. Richter [519] pays close attention to some details omitted in [188] and [238]. Gel'fand, Graev, and Shapiro [193] have analogous range results on $\mathbb{C}^n$ (compare, Grinberg [237]). Gonzalez [224] derives Richter's form of theorem 4.31 using a direct group-theoretic method rather than local coordinate calculations. It is hard to describe this operator in a succinct way, so the reader is referred to [223] for the details. For similar reasons we do not give the details for [188], [519], and [224].

Carton-Lebrun's paper [79], mentioned earlier, not only yields a simplified proof of the Helgason range theorem 4.29, but also gives a sufficient condition for a function to be the image under the Radon transform of a continuous L^1 function.

The Radon transform can be extended to distributions. Theorem 3.29 on the formal adjoint of the k -plane transform states that $\langle f, P^\# g \rangle_{\mathbb{R}^n} = \langle Pf, g \rangle_{G_{k,n}}$. By allowing f to be a distribution on $\mathbb{R}^n$ and g to be a test function, one can define the k -plane transform of a distribution. Dually, by allowing g to be a distribution on $\mathcal{G}_{k,n}$ and f to be a test function on $\mathbb{R}^n$, one can define the formal adjoint $P^\# g$ of a distribution g . Gel'fand, Graev, and Vilenkin [194] and Ludwig [405] show that the resulting functionals are continuous on various test spaces and, thereby, the k -plane transform of tempered distributions and compactly supported distributions are well defined.

Ludwig [405] presents some interesting mapping properties of the Radon transform on both functions and distributions. Hertle [302] contains some interesting extensions to the Helgason range theorem and Ludwig's results. In these theorems, $\mathcal{D}'$ denotes the space of distributions, $\mathcal{E}'$ denotes the space of compactly supported distributions, and $\mathcal{S}'$ denotes the space of tempered distributions.

Theorem 4.33 (Hertle [302]). *The formal adjoint of the Radon transform has the following behavior:*

$$R^\# : \mathcal{S}'(S^{n-1} \times \mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is surjective}$$

$$R^\# : \mathcal{C}^\infty(S^{n-1} \times \mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \text{ is surjective}$$

$$R^\# : \mathcal{D}'(S^{n-1} \times \mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R}^n) \text{ is not surjective.}$$

Remark 4.34. Gonzalez [224] contains the following generalization of the second part of this theorem: if P is the k -plane transform then $P^\# : \mathcal{C}^\infty(\mathcal{G}_{k,n}) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ is surjective.

Theorem 4.35 (Hertle [302]). *The Radon transform on $\mathbb{R}^n$ has the following behavior:*

$$R : \mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{E}'(S^{n-1} \times \mathbb{R}) \text{ is a topological isomorphism}$$

$$R : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}(S^{n-1} \times \mathbb{R}) \text{ has no continuous inverse.}$$

Finally, Hertle describes the structure of some of these spaces.

Theorem 4.36 (Hertle [302]). *$R(\mathcal{E}'(\mathbb{R}^n))$ is a reflexive nuclear DF space. $R(\mathcal{D}(\mathbb{R}^n))$ is not a DF space, is not topologically complemented in $\mathcal{D}(S^{n-1} \times \mathbb{R})$, and is neither bornological nor barreled.*

Hertle [300] has some interesting topological vector space properties of the Radon transform restricted to the unit ball. He shows that the maps

$$R : L^1(B(1)) \rightarrow L^1(S^{n-1} \times [-1, 1]) \text{ and}$$

$$R : C(B(1)) \rightarrow C(S^{n-1} \times [-1, 1])$$

are continuous, but not weakly compact. However

$$R : L^p(B(1)) \rightarrow L^p(S^{n-1} \times [-1, 1])$$

is compact for $1 < p < \infty$.

The reader is referred to Hertle [300] for the proofs and other details.

4.3.2 Partial Differential Equations and the Radon Transform

A close relationship exists between the Radon transform and the study of partial differential equations. For example, John's theorem, theorem 4.1, shows that solutions to certain partial differential equations can be found from the range of a k -plane transform. However, we can only give a brief survey of this interesting topic. The reader is referred to John [324, 325], Ludwig [405], Lax and Phillips [378], and Helgason [291] for further results and details.

It is well known that certain partial differential equations can be solved by superposition of trigonometric functions (John [327], Dym and McKean [138]). This is done by transforming the partial differential equation into an ordinary one via the Fourier transform. Analogously, certain partial differential equations can be solved by superposition of plane waves. A **plane wave**, more precisely, a **plane wave orthogonal to a direction** $\theta \in S^{n-1}$ is a function of the form $u(x) = f(\langle x, \theta \rangle)$, where f is a function defined on $\mathbb{R}$. From this definition, it is obvious that a plane wave orthogonal to θ is constant on each hyperplane orthogonal to θ , because as we know from chapter 2 each such hyperplane is of the form $\langle x, \theta \rangle = p$. For this reason, the Radon transform may play a role in finding solutions to partial differential equations which are superpositions of plane waves.

We illustrate the idea with one example derived from Helgason [291]. The reader can refer to John [325], Helgason [291], and Dym and McKean [138] for more details and examples. We solve the Cauchy problem for the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \quad (4.27)$$

$$u(x, 0) = f(x), \quad (4.28)$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \quad (4.29)$$

with $f, g \in C_0^\infty(\mathbb{R}^n)$.

The solution involves functions of the form $g(\theta, p)$ defined on $S^{n-1} \times \mathbb{R}$. The Hilbert transform applied to such a function is applied to the second variable with θ fixed. We let ∂ be the partial derivative acting with respect to the variable p . We also define K to be i times the Hilbert transform, with respect to the p variable, if n is even, and to be the identity operator if n is odd. Theorem 3.53, section 3.6.1 can be expressed as

$$f(x) = \frac{1}{2}(2\pi i)^{1-n} \int_{S^{n-1}} K \partial^{n-1} Rf(\theta, \langle x, \theta \rangle) d\theta.$$

Theorem 4.37. *The solution to the Cauchy problem for the wave equation (4.27)–(4.29) is given by*

$$u(x, t) = c \int_{S^{n-1}} K \left(\partial^{n-1} Rf + \partial^{n-2} Rg \right) (\theta, \langle \theta, x \rangle + t) d\theta \quad (4.30)$$

where the constant c is given by

$$c = \frac{1}{2} (2\pi i)^{1-n}.$$

Proof. Let $Q = \frac{\partial^2}{\partial t^2} - \Delta$.

Because of the regularity of the functions, the differential operator Q can be brought inside the integral of equation (4.30). Once inside, the integrand is seen to be of the

form $v(x, t) = h(\theta, \langle \theta, x \rangle + t)$, where $h(\theta, x)$ is the value of the function $K(\partial^{n-1} Rf + \partial^{n-2} Rg)$ evaluated at (θ, x) .

But for any $h \in C^2(\mathbb{R})$, for fixed θ , the function $v(x, t) = h(\langle \theta, x \rangle + t)$ satisfies $Q(v) = 0$. This follows from the easily verified equations

$$\begin{aligned}\frac{\partial^2}{\partial t^2} h(\langle \theta, x \rangle + t) &= h''(\langle \theta, x \rangle + t) \\ \frac{\partial^2}{\partial x_j^2} h(\langle \theta, x \rangle + t) &= \theta_j^2 h''(\langle \theta, x \rangle + t).\end{aligned}$$

From these equations and the fact that $|\theta| = 1$ we see that the Laplacian of $h(\langle \theta, x \rangle + t)$ and $\frac{\partial^2}{\partial t^2} h(\langle \theta, x \rangle + t)$ both have a common value and hence that $Q(v) = 0$.

This shows that the integrand of the integral defining $Q(u)$ is identically zero. Hence $Q(u) = 0$ and u satisfies the wave equation (4.27).

It remains to prove that equations (4.28) and (4.29) are satisfied. Let us first treat the case where n is odd. In this case K is the identity operator. Also, $Rg(\theta, \langle \theta, x \rangle)$ is an even function of θ ; this follows from the known evenness of the Radon transform. Because n and hence $n - 2$ is odd, then $\partial^{n-2} Rg$ is odd and integrating it over the sphere gives zero. Hence

$$\begin{aligned}u(x, 0) &= c \int_{S^{n-1}} K \left(\partial^{n-1} Rf + \partial^{n-2} Rg \right) (\theta, \langle \theta, x \rangle) d\theta \\ &= c \int_{S^{n-1}} \partial^{n-1} Rf (\theta, \langle \theta, x \rangle) d\theta \\ &= f(x)\end{aligned}$$

since the second line of this equation is exactly the right-hand side of the inversion formula for the Radon transform given by theorem 3.53. This proves equation (4.28). Equation (4.29) is proved in the same way:

$$\begin{aligned}\frac{\partial u}{\partial t}(x, 0) &= c \int_{S^{n-1}} K \left(\partial^n Rf + \partial^{n-1} Rg \right) (\theta, \langle \theta, x \rangle) d\theta \\ &= c \int_{S^{n-1}} \partial^{n-1} Rg (\theta, \langle \theta, x \rangle) d\theta \\ &= g(x)\end{aligned}$$

since now $\partial^n Rf$ is odd.

If n is even, then K is equal to iH where H the Hilbert transform and we get, again by theorem 3.53, that

$$\begin{aligned} u(x, 0) &= c \int_{S^{n-1}} K \left(\partial^{n-1} Rf + \partial^{n-2} Rg \right) (\theta, \langle \theta, x \rangle) d\theta \\ &= c \int_{S^{n-1}} K \left(\partial^{n-1} Rf \right) (\theta, \langle \theta, x \rangle) d\theta \\ &= f(x) \end{aligned}$$

since the Hilbert transform interchanges odd and even functions, so in this case $K \partial^{n-2} Rg$ is odd. This proves equation (4.28) and a similar calculation proves equation (4.29). ■

Remark 4.38. Helgason's book [291] has quite a few more interesting details about the relation of the Radon transform to partial differential equations.

Our proof of theorem 4.37 is essentially the same as the proof of theorem 7.3 in Helgason's book on the Radon transform, second edition [291]. This edition of Helgason's book has some misprints related to the theorem. In equation (96), page 42 of [291], $u_t(x, 0) = f_1(0)$ should read $u_t(x, 0) = f_1(x)$ and the displayed equation in lemma 7.2 should read $v(x, t) = h(\langle x, \omega \rangle + t)$ not $v(x, t) = h(\langle x, \omega \rangle)$.

Remark 4.39. The following idea is hidden in the details of the proof: when applicable, the Radon transform reduces a partial differential equation to an ordinary differential equation with a parameter, the parameter being the direction θ .

4.4 Applications of the Consistency Conditions for the Range of the Radon Transform

Any square integrable function can be expanded in a spherical harmonic series. We briefly summarize some facts about spherical harmonics: details may be found in Stein and Weiss [583] or Seeley [549]. A spherical harmonic is the restriction of a homogeneous harmonic polynomial to the unit sphere. A harmonic polynomial is a polynomial that satisfies the Laplace equation $\Delta f = 0$.

The set of spherical harmonics form a dense linear subspace of $L^2(S^{n-1})$. Spherical harmonics of different degrees are orthogonal with respect to the inner product on the Hilbert space $L^2(S^{n-1})$. Also the spherical harmonics of a specific degree l form a finite dimensional Hilbert space $\mathcal{Y}_l$ of dimension $N(n, l)$. It is possible to compute the exact value of $N(n, l)$, but this is not necessary for our discussion. We choose an orthonormal basis $\{Y_{jl} : j = 1, \dots, N(n, l)\}$ of $\mathcal{Y}_l$. For later use note that the homogeneity of spherical harmonics implies that the parity of Y_{jl} is the same as the parity of l .

As a consequence of this discussion, any L^2 function on the sphere can be expanded in a series $\sum_{l=0}^{\infty} \sum_j c_{jl} Y_{jl}(\theta)$ where $j = 1, \dots, N(n, l)$. A standard separation of variables argument applied to the polar coordinate representation $x = r\theta$ then shows

that any function $f \in L^2(\mathbb{R}^n)$ can be expanded in a series in the following way. Let $\{b_k\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$. Then there are coefficients $a_{kjl} \in \mathbb{C}$ such that

$$f(r\theta) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_j a_{kjl} b_k(r) Y_{jl}(\theta).$$

Similar reasoning applies to functions in the range of the Radon transform of L^2 functions on the unit ball of $\mathbb{R}^n$. Such functions are contained in $L^2(S^{n-1} \times [-1, 1])$ and we get the expansion

$$Rf(\theta, t) = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \sum_{j=1}^{N(n,l)} c_{kjl} L_k(t) Y_{jl}(\theta) \quad (4.31)$$

where L_k is the Legendre polynomial of degree k . It is known that L_k is even if k is even and odd if k is odd. The Legendre functions form an orthonormal basis of $L^2([-1, 1])$ and L_k is orthogonal to any polynomial of degree $< k$ on $[-1, 1]$ [see the discussion of Legendre polynomials following equation (4.15)].

By the orthonormality of L_k and Y_{jl} we obtain the following integral formula for the coefficients

$$c_{kjl} = \int_{S^{n-1}} \int_{-1}^1 Rf(\theta, t) L_k(t) Y_{jl}(\theta) dt d\theta.$$

We now apply the consistency conditions: by theorem 4.24 we know that for all $k \in \mathbb{Z}^+$, the function of $\theta \in S^{n-1}$ defined by $\int_{-\infty}^{\infty} t^k Rf(\theta, t) dt$ is the restriction to the sphere of a homogeneous polynomial of degree k . Therefore the function $p_k(\theta) = \int_{-1}^1 L_k(t) Rf(\theta, t) dt$ is the sum of $k+1$ homogeneous polynomials of degree $0, 1, \dots, k$ since the degree of L_k is k . We can rewrite the coefficients c_{kjl} as

$$c_{kjl} = \int_{S^{n-1}} p_k(\theta) Y_{jl}(\theta) d\theta. \quad (4.32)$$

Therefore by orthogonality of the spherical harmonics $Y_{jl}(\theta)$ we will have $c_{kjl} = 0$ if $l > k$.

Furthermore, we can prove that $p_k(\theta)$ is an even function of θ for k even and an odd function for k odd: use the change of variable $s = -t$, and the evenness of the Radon transform (theorem 2.39) to get

$$\begin{aligned} p_k(-\theta) &= \int_{-1}^1 L_k(t) Rf(-\theta, t) dt = \int_{-1}^1 L_k(-s) Rf(-\theta, -s) ds \\ &= \pm \int_{-1}^1 L_k(s) Rf(\theta, s) ds = \pm p_k(\theta), \end{aligned}$$

the parity of the sign being the same as the parity of k . We already observed that $Y_{jl}(\theta)$ has the same parity as l . Therefore, the integrand $p_k(\theta)Y_{jl}(\theta)$ has odd parity any time k and l have opposite parity and even parity any time they have the same parity. As a consequence $c_{kjl} = 0$ whenever k and l have opposite parity since equation (4.32) shows that c_{kjl} is the integral of an odd function over the sphere in this case. Observing that k and l have the same parity if and only if $k + l$ is even, then the expansion (4.31) simplifies to

$$Rf(\theta, t) = \sum_{k=1}^{\infty} \sum_{\substack{l=0 \\ k+l \text{ even}}}^k \sum_{j=1}^{N(n,l)} c_{kjl} L_k(t) Y_{jl}(\theta). \quad (4.33)$$

The Legendre polynomials are a special case of the Gegenbauer polynomials C_k^λ . The Gegenbauer polynomials C_k^λ of fixed index λ are orthogonal polynomials on a weighted L^2 space on $[-1, 1]$. The weight depends on λ , see Natterer [444], section VII.3. If $\lambda = \frac{1}{2}$, then the Gegenbauer polynomials become the Legendre polynomials and the weight is identically 1. We developed formula (4.33) using Legendre polynomials because it was the simplest case. The general case is similar, but we must multiply by the weight function.

Louis [399], building on the two-dimensional results of Cormack [101], Marr [425], and Louis [397], is able to relate the spherical harmonic expansion of $f \in L^2(\mathbb{R}^n)$ to the spherical harmonic expansion of its Radon transform, as in equation (4.33). This is a generalization of a result of Ludwig [405].

As a result of this relation Louis [399] gives an inversion formula for the Radon transform and a characterization of the null space of the Radon transform restricted to finitely many directions. Louis [398] continues this theme by characterizing the frequency distribution of this type of null space. Louis uses the term “ghost” for an element of this null space, because such elements appear to be tumors in medical CT applications, but are actually just artifacts of the inversion of the Radon transform. Logan [393] and Maass [410] have related results about the frequency distribution of ghosts of the x-ray transform. Without going into detail, it turns out that ghosts have most of their energy in the high-frequency components of their Fourier transforms, if a sufficiently large number of projections is used. More details may be found in chapter 5, section 5.9.

Another area of interest concerns attempts to use range conditions for the Radon transform, such as the moment conditions or John’s ultrahyperbolic systems of partial differential equations, to improve the quality of practical CT reconstructions. Suppose that we have built some sort of scanner that attempts to reconstruct a function f from its Radon transform. For example we can think of a CT scanner for which the Radon transform is approximated by observing the behavior of x-rays passing through f . In practice one only observes $Rf(\theta, p)$ for a finite number of directions θ and for a finite number of sample points p and those observations are usually corrupted with some amount of noise. However, Smith, Solmon, and Wagner [566] showed that there is always some function h that has projections matching our observed data at the prescribed directions and sample points (see [566], page 1269 for the

precise statement). Unfortunately, this function h may have no relation to the original function f .

Therefore, there seems to be some hope that if we could somehow modify the observed projections so that they exactly matched the corresponding projections given by Rf , then the resulting reconstruction g would be better than that obtained without this process. Patch [469] shows that in the absence of a priori information about f , any such attempt would be futile, provided that filtered backprojection was the reconstruction method. Here is an informal overview of her argument.

Assume that f is a compactly supported L^2 function which lives in the unit ball of $\mathbb{R}^n$. Let g be the function that is obtained from the observed projection data. Although the observed data are finite, by interpolation we can assume that g is defined everywhere on $S^{n-1} \times [-1, 1]$, is zero elsewhere, and is L^2 . We could call g “the function obtained from the observed data after interpolation,” but we refer to g simply as the “observed function.” It is known that $Rf \in L^2(S^{n-1} \times [-1, 1])$, so if we define $\varepsilon = g - Rf$, then ε is also in $L^2(S^{n-1} \times [-1, 1])$. We think of ε as the error between the Radon transform of the original function f and the observed function g . We further define ε_0 as the projection of ε on the range $H = R(L^2(B(1)))$ of the Radon transform restricted to $L^2(B(1))$ and we define ε_1 as the projection onto $H^\perp$. We can now analyze a filtered backprojection reconstruction of f from g by using the inversion formula $c\Lambda R^\# R(f) = f$ as an abstract representation of the filtered backprojection process:

$$c\Lambda R^\#(g) = c\Lambda R^\# R(f) + c\Lambda R^\#(\varepsilon_0) + c\Lambda R^\#(\varepsilon_1)$$

But the orthogonal complement of the range of the Radon transform $H^\perp$ is contained in the null space of the adjoint $R^\#$, so $\Lambda R^\#(\varepsilon_1) = 0$, whereas $\Lambda R^\# R(f) = f$ by the inversion formula for the Radon transform. Hence, we get the approximation

$$f = c\Lambda R^\#(g) - c\Lambda R^\#(\varepsilon_0). \quad (4.34)$$

Note that this approximation was obtained without any processing to ensure that g was in the range of the Radon transform. If we did such processing and arrived at a function $g_1 \in H$, then the only difference in the preceding computation would be that ε_1 would be the zero function. We would again arrive at the approximation (4.34). Therefore, there would be no improvement in the reconstruction process by processing g to ensure that it would lie in the range of the Radon transform.

Nonetheless, Patch showed that certain defects in the detectors of CT scanners could be found by looking for violations in the moment conditions characterizing the range of the Radon transform. Also range conditions could help when using other algorithms. For example Defrise, Noo, and Kudo [126] describe a method to improve the accuracy of rebinning algorithms for helical computerized tomography with multirow detectors. The authors state: “In theory, exact rebinning could be achieved by solving John’s partial differential equation to virtually move the x-ray source . . . we do not attempt to solve John’s equation exactly. Instead, we use John’s equation to compute a first order correction to the rebinning algorithm. Tests with simulated data demonstrate a significant improvement of image quality, obtained with a negligible increase of the computation time and of the sensitivity to noise.”

4.5 Additional References and Results

The books John [325], Natterer [444], Helgason [291], Gel'fand, Gindikin, and Graev [189], and Ehrenpreis [149] are good general sources for the subject of ranges of the Radon transform or the relation of the Radon transform to partial differential equations.

4.5.1 Range Theorems

Lax and Phillips [378] prove a range theorem for the Radon transform and use it to derive some results on partial differential equations.

Peters [474] shows how to determine the convex hull of the support of a continuous L^1 function on $\mathbb{C}^n$ from Rf . He also gives an example in certain $\mathbb{R}^n$ showing that the existence of a finite number of derivatives of Rf is not sufficient to prove the continuity of f .

Let $\{\theta_1, \dots, \theta_k\}$ be a collection of distinct unit vectors such that no pair point in opposite directions. Falconer [155] defines a set X to be a normal convex set in $\mathbb{R}^n$ with respect to the unit vectors $\{\theta_1, \dots, \theta_k\}$ if X is a proper, compact, convex subset of $\mathbb{R}^n$ such that for any $x \in X$, there is at most one index i such that the $(n-1)$ -dimensional Lebesgue measure of X intersected with the hyperplane orthogonal to θ_i at the distance $p = \langle x, \theta_i \rangle$ from the origin is zero. In particular, any smooth compact convex set is proper with respect to any finite set of unit vectors (only the unit vector orthogonal to the tangent hyperplane at x can yield a zero volume intersection with X). Falconer then proves a range theorem characterizing functions $G(\theta, p)$ which are projections of some $f \in L^q(X)$, $1 < q \leq 2$. These conditions are analogous to the consistency conditions (theorem 4.29), including the moment condition, but they apply to L^q functions on normal convex sets.

Droste [130] gives an elementary range characterization of the Radon transform of functions supported in a ball. His proof is based on the slice-projection theorem and the Paley–Wiener theorem for the Fourier transform.

Solmon [571] has some interesting extensions of the range theorem (theorem 4.29). First, note that the range theorem characterizes the image of rapidly decreasing functions under the Radon transform. Solmon investigates the image of certain C^∞ functions, which may not be rapidly decreasing. One of his main results is that any even function $g \in \mathcal{S}(S^{n-1} \times \mathbb{R})$ is the image, under the Radon transform, of a C^∞ function f satisfying the growth condition $f \in O(|x|^{-n})$. This is interesting because no moment conditions, of the form $P_j(\theta) = \int_{-\infty}^{\infty} g(\theta, s)s^j ds$ is a homogeneous polynomial on the sphere, are required. Furthermore, Solmon shows that a C^∞ function f such that $Rf \in \mathcal{S}(S^{n-1} \times \mathbb{R})$ satisfies the growth condition $f \in O(|x|^{-n-m-1})$ if and only if $P_j(\theta)$ is homogeneous polynomial of degree j for each $j = 0, 1, \dots, m$. An interesting consequence of his main result is that g satisfies the $j = 0$ moment condition if and only if $f \in L^1 \cap C^\infty$. Solmon also develops asymptotic estimates and an inversion formula for the backprojection operator (compare, Madych and Solmon [417] and Peters [478]).

Louis and Rieder [403] derive a characterization of the range of the fan beam transform. They apply this result to limited view fan beam tomography by computing an approximation to the complete set of fan beam projections from the given limited views.

Bray and Solmon [68] characterize the range of a generalized Radon transform. This transform is defined by integrating functions over horocycles on a semisimple symmetric space of noncompact type.

Znamenskiĭ [628] uses a form of tomography on analytic functionals to characterize compact subsets of $\mathbb{C}^n$ whose holomorphically convex hulls have simply connected intersections with complex lines. These sets are precisely the ones satisfying a certain consistency condition involving a homogeneous polynomial related to the functionals.

Katsevich [340] establishes range theorems for the Radon transform and its dual for functions and distributions with bounded rate of decay at ∞ (compare, Katsevich [335] and [339]).

Berenstein and Casadio Tarabusi [44] give a characterization of the range of the k -plane transform on real hyperbolic spaces in terms of a system of second-order partial differential equations or as a single fourth-order equation. They continue this study in [45] where they show that the range of the k -plane transform can also be characterized in real hyperbolic spaces by moment conditions. This is in contrast to the situation for the k -plane transform on $\mathbb{R}^n$, where moment conditions do not suffice to characterize the range of the k -plane transform for $k < n - 1$ (see Gonzalez [223]).

Takehi [329] characterizes the range of the k -plane transform on complex projective n space for $1 \leq k \leq n - 2$. The range is the kernel of a sum of a system of second-order ultrahyperbolic differential operators. Gonzalez [225] has analogous results for real projective spaces. In [331] Takehi and Tsukamoto characterize the range of the Radon transform on real and complex projective spaces in terms of a differential operator with values in the sections of a certain vector bundle. Takehi [330] considers the spherical Radon transform which integrates C^∞ functions over k -dimensional spheres in S^{n-1} . He shows that the range of this transform is the kernel of a fourth order differential operator. See also Abouelaz and Daher [3] and Abouelaz [2].

Rubin [530] gives range theorems for the Radon transform restricted to $L^p(\mathbb{R}^n)$, $1 < p < \frac{n}{n-1}$.

Symeonidis [591] obtains range theorems for the k -plane transform $Pf(\eta, x')$, where η is restricted to the orbit of a matrix group.

Suppose that we are given compactly supported functions ρ_j defined on $\mathbb{R}$ and we know that each such function is the exact x-ray projection $R_{\theta_j}f$ of some fixed but unknown compactly supported function defined on $\mathbb{R}^2$ for some finite number of fixed but unknown directions $\theta_j \in S^1$. Let us call such a collection of functions ρ_j a data set. Basu and Bresler [38] use the Helgason–Ludwig consistency conditions to show that for almost every such data set with at least nine elements the directions θ_j can be recovered up to an orthogonal transformation, that is, the angles of these directions can be recovered up to an additive constant angle.

See section 5.10.2 for references on range theorems for the attenuated and exponential Radon transforms.

4.5.2 Differential Equations

Lax and Phillips [379] develop a scattering theory for certain first-order linear systems of partial differential equations. They use the Radon transform to derive some of the results. In [381] they develop a scattering theory for a continuous semigroup of contractions in Hilbert space and apply that along with the Radon transform to the scattering theory of the acoustic equation with dissipative boundary conditions in $\mathbb{R}^n$ with n odd. Results for the acoustic equation in even dimensional spaces is given in [380]. Pyž'janov [497] uses the Lax and Phillips technique involving the Radon transform to derive a representation of the scattering operator and its inverse for Maxwell's equations in an exterior domain. Lax and Phillips [382] use the Radon transform to develop harmonic analysis and scattering theory for Riemannian symmetric spaces.

Bardos [36] uses the Radon transform to study the asymptotic behavior of a first-order linear partial differential equation.

Helgason [275] uses the Radon transform, among other powerful techniques, to show that if D is an invariant differential operator on a symmetric space of noncompact type, then the equation $Du = f$ has a C^∞ solution for any compactly supported C^∞ function f . He extends this result in [274].

Helgason [265] initiated the study of differential operators on homogenous spaces (compare, Helgason [266, 269] and Semjanistyi [550]).

Gindikin [197, 198] develops a Radon transform on the torus in which integration is over closed totally geodesic hyperplanes. He applies this to finding integral formulas for holomorphic functions, the existence of fundamental solutions of elliptic equations on the torus and a solution of the Cauchy problem for hyperbolic equations with periodic initial data.

Činačev and Osovs'kiĭ [94] use both the Laplace and Radon transforms to determine frequency characteristics of partial differential equations.

Willis [622] applies the Radon transform to self-similar mixed boundary value problems of elastodynamics.

Eršov and Šihov [152] apply the Radon transform in the course of establishing an integral representation of the neutron density in phase space for a certain neutron transport equation. Newton [448] uses the inverse Radon transform to generate an integrodifferential equation that has soliton solutions vanishing at large distances. For further references on the relation between scattering theory for the transport equation and tomography consult Hejtmanek [264].

Guillemin and Schaeffer [247] characterize wave-front sets and Fourier integral operators using the Radon inversion formula (compare, Guillemin and Sternberg [248], chapter 6).

Chen [87] applies the Radon transform to the study of the wave equation.

Cormack and Quinto [107] use the Radon transform to show that smooth solutions to the Darboux partial differential equation are determined by the data on any characteristic cone with vertex on the initial surface. They also show that if the data are an entire function, then a real analytic solution exists (compare, Rhee [517]).

Barrett [37] defines the “dipole-sheet” transform by composing the differential operator $\frac{1}{p} \frac{\partial}{\partial p}$ with the Radon transform on $\mathbb{R}^3$. He applies this transform to the Helmholtz

partial differential equation, thereby reducing to a two-parameter family of ordinary differential equations.

The Clifford algebra $\mathbf{C}_m$ is the algebra generated over $\mathbb{C}$ with generators $e_1, \dots, e_m$ with the relations $e_i e_j - e_j e_i = 2\delta_{ij}$. A Clifford algebra valued function $f : \mathbb{R}^{m+1} \rightarrow \mathbf{C}_m$ is said to be left monogenic if it satisfies the partial differential equation

$$\left(\frac{\partial}{\partial x_0} + \sum_{j=1}^m \frac{\partial}{\partial x_j} e_j \right) f = 0. \quad (4.35)$$

This equation is analogous to the Cauchy–Riemann equations because it is known that the left monogenic functions satisfy many function theoretic properties possessed by holomorphic functions. The solutions to the system (4.35) form a right $\mathbf{C}_m$ module. Without going into too many details, a hypercomplex functional is defined to be an element of the dual module of the module of monogenic functions defined on an open subset of $\mathbb{R}^{m+1}$.

Sommen [575]–[578] studies a Radon transform acting on hypercomplex functionals. This Radon transform is studied by using plane wave decompositions of solutions to the generalized Cauchy–Riemann equations (4.35). Sommen is able to find an inversion formula for this transform. He also characterizes the range of this transform. Elements of the range satisfy a system of partial differential equations closely related to the generalized Cauchy–Riemann system (4.35).

The following question is reminiscent of problems in tomography: Can a force of finite extent and duration be found from the subsequent disturbance it makes? From the mathematical point of view this is the same as asking whether solutions to the hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = Q \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) u$$

can be found if one knows u for values of t for which the sets $\mathbb{R}^n \times \{t\}$ are disjoint from a compact domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$. Here Q is a polynomial and the domain Ω is the region in space and time which generates the subsequent “disturbances” $u(x, t)$ for $(x, t) \notin \Omega$. Hamaker [253] investigates this question. Let $P = \frac{\partial^2}{\partial t^2} - Q(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. Let f be in $L_0^2(\Omega)$ and assume that u is a solution that determines the values of f on Ω . If v has support in Ω , then $Pv = 0$ outside Ω , whereas u coincides with $u + v$ there, so if f could be determined from u outside Ω , then $f + Pv$ could be also. Therefore, Hamaker considers equivalence classes $[f] = \{f + Pv : v \in L_0^2(\Omega)\}$ and he proves the following theorem:

Theorem 1.1 [253]. *Let Ω be a relatively compact, open, convex subset of $\mathbb{R}^n \times \mathbb{R}$ and suppose that $Pu = f$ for some $u = u(x; t)$ which vanishes for large negative values of t . Then for large enough T , a representative of the class $[f]$ can be computed from the Cauchy data for u on $\mathbb{R}^n \times \{t\}$.*

The proof involves taking the Radon projection R_θ of the equation $Pu = f$. This results in an ordinary differential equation relating $R_\theta u$ to $R_\theta f$. This equation can be solved and then the inverse Radon transform can be used to prove theorem 1.1.

Burridge and Beylkin [73] derive some relations on differential forms arising in certain double integrals on the sphere S^{n-1} . They have several applications, including a new inversion formula for the Radon transform on $\mathbb{R}^n$ and an application to inverse scattering theory.

Solmon [572] characterizes the null space of a divergent beam x-ray transform with a finite source set using a certain partial differential equation. For more details see the discussion of [572] in section 5.10.2.

Branson, Ólafsson, and Schlichtkrull [66] obtain a proof of the Huygens principle and equipartition of energy for the Dirac and Maxwell equations via their theory of the Fourier and Radon transform of vector bundle sections over certain symmetric spaces.

Abouelaz [1] studies a generalized Radon transform on the special orthogonal group with an application to the Cauchy problem for the wave equation on a sphere.

Given a differentiable manifold X , a $\mathcal{D}_X$ module is a sheaf of modules over the sheaf of rings of germs of differential operators on X . See section 5.8 for a short discussion on $\mathcal{D}$ modules and their relation to the Radon transform.

Generalizations and Variants of the Radon Transform

5.1 Introduction

There are two main ways of generalizing the Radon transform. One can add a weight to the hyperplane integral or one can integrate over subvarieties that are not linear. The first idea leads to the *generalized Radon transform in $\mathbb{R}^n$* that we study in section 5.4. An example of a generalized Radon transform is the attenuated Radon transform. This transform is of practical importance because it models single-photon emission computed tomography (SPECT). We investigate the attenuated Radon transform in section 5.3 along with the exponential Radon transform that is also used in SPECT. In this section we also mention positron emission tomography (PET), although this version of tomography reduces to the standard Radon transform.

Another way of generalizing the Radon transform is to integrate over more general submanifolds of $\mathbb{R}^n$ than hyperplanes and k planes. More generally, we can integrate over submanifolds of a manifold X that is more general than $\mathbb{R}^n$. For example, Funk in his 1916 paper [184] showed that an even function on a sphere is completely determined by its integrals over great circles. This was a year before Radon's paper [508] appeared. The transform that sends a function on the unit sphere to its integrals over great circles is now called the Funk transform or the spherical Radon transform. A major advance in studying Radon transforms over nonlinear varieties was created in 1964 by Helgason [267] and we introduce this idea in section 5.5.

The divergent beam transform, a relative of the Radon transform is introduced and studied in section 5.2. It is important as a model of fan beam tomography and also for its use in the attenuated Radon transform.

Sections 5.6, 5.7, and 5.8 give brief descriptions of the relation of the Radon transform to twistors and the Penrose transform, $\bar{\partial}$ cohomology in several complex variables and the theory of $\mathcal{D}$ modules, respectively.

In section 5.9 we look at a final variant of the Radon transform called the finite Radon transform. This arises by taking x-ray projections in finitely many directions.

The prerequisites for reading this chapter include those for chapters 2 and 3. We also make use of the Hardy–Littlewood–Sobolev theorem on fractional integration,

which was described in chapter 3, and the elementary theory of Riesz transforms and conjugate harmonic functions. The applications of these more advanced ideas are carefully described and referenced.

In this chapter we use the term invertibility in a nonstandard way. We say that an operator is invertible if it is injective. We distinguish this from the ability to actually calculate the inverse, that is to say, to obtain an explicit inversion formula.

5.2 Divergent Beam and Cone Beam Transforms

In early CT scanners x-ray beams were projected parallel to each other for each direction θ . This is modeled by the Radon transform and is called the *parallel beam geometry*. However, it is more efficient to emit x-rays from multiple sources with each source emitting x-rays in many different directions. Because the x-rays diverge in multiple directions from a single source, this is called the *divergent beam geometry*. In dimension $n = 2$ it is called the *fan beam geometry*, and in dimension $n = 3$ it is called the *cone beam geometry*. The reason for this terminology is apparent from the accompanying figure. Most modern CT scanners use either the fan beam or cone beam geometry, although a helical beam geometry is becoming more popular as well.

The difference between the parallel beam geometry and the divergent beam geometry is illustrated in figure 5.1. The diagram on the left shows the parallel beam geometry which is modeled by the Radon transform. The diagram on the right illustrates divergent beam geometry. The transform that models the divergent beam geometry is called the *divergent beam transform* and is defined as follows.

Definition 5.1. The *divergent beam transform* $\mathcal{D}$ of a function f on $\mathbb{R}^n$ is defined for $\theta \in S^{n-1}$ and $x \in \mathbb{R}^n$ by

$$\mathcal{D}f(x, \theta) = \int_0^\infty f(x + s\theta) ds$$

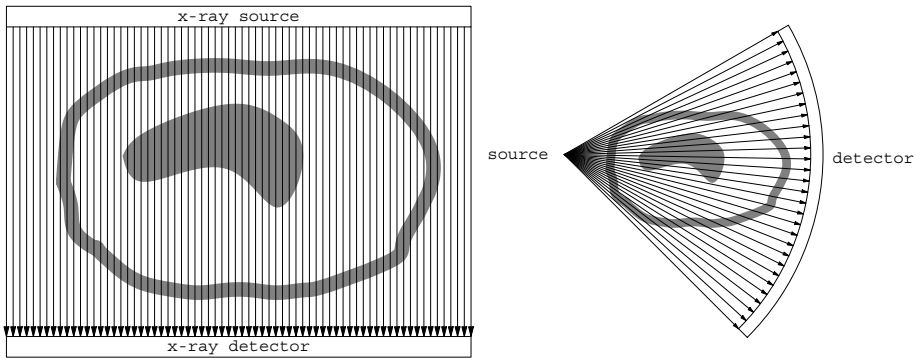


Figure 5.1. X-ray scanning geometries. (Left) Parallel beam geometry. (Right) Divergent beam geometry.

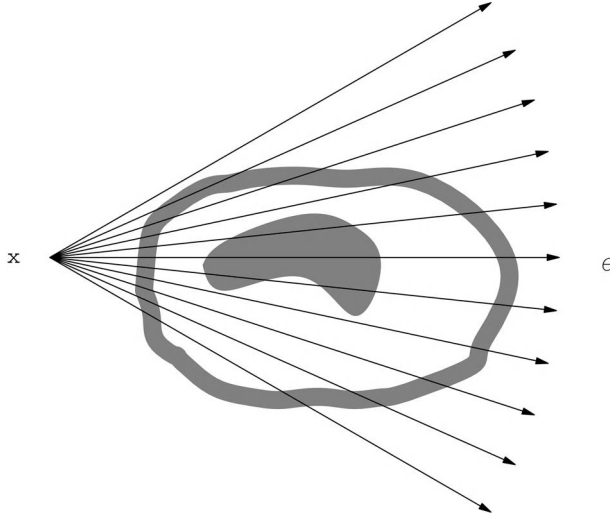


Figure 5.2. Fan beam geometry.

For fixed $x \in \mathbb{R}^n$ we also define the operator $\mathcal{D}_x$ by the equation

$$\mathcal{D}_x f(\theta) = \mathcal{D}f(x, \theta)$$

We call $\mathcal{D}$ a **restricted divergent beam transform with domain A** , if the domain of $\mathcal{D}f$ is a proper subset A of $S^{n-1} \times \mathbb{R}^n$. This corresponds to knowing x-rays emanating from a restricted set of points x in $\mathbb{R}^n$ and a restricted set of directions θ in S^{n-1} .

Closely related to the divergent beam transform is the **line integral transform**, which is defined as follows: for a function f on $\mathbb{R}^n$, for $\theta \in S^{n-1}$, and for $x \in \mathbb{R}^n$ define

$$\mathcal{L}f(x, \theta) = \mathcal{L}_x f(\theta) = \int_{-\infty}^{\infty} f(x + s\theta) ds$$

Except for notation the line integral transform is the same as the x-ray transform; (i.e., the 1-plane transform; compare, definition 3.16, section 3.4). It is clear that

$$\mathcal{L}_x f(\theta) = \mathcal{D}_x f(\theta) + \mathcal{D}_x f(-\theta)$$

Figure 5.2 helps to explain the connection between the divergent beam transform and the fan beam geometry. In figure 5.2 we have shown an object, which we will denote by f , and an x-ray source located at the point x . We have also shown a few of the many x-ray beams emanating from x . We assume that there are x-ray detectors positioned somewhere to the right of the figure. We have also indicated a particular direction θ . Fix this direction for a moment.

According to the Lambert-Beer law (equation (1.5), section 1.9), if $L(\theta)$ represents the ray emanating from the source in the direction θ , if $N_0(\theta)$ represents the the number of photons emitted from the source and traveling in the direction θ , and if $N_1(\theta)$

represents the number of these photons recorded by the detector, then

$$\int_{L(\theta)} f(t) dt = \log \left(\frac{N_0(\theta)}{N_1(\theta)} \right) \quad (5.1)$$

The ray $L(\theta)$ is parametrized by $t = x + s\theta$, $s \in (0, \infty)$. Therefore, equation (5.1) becomes $\int_0^\infty f(x + s\theta) ds = \mathcal{D}f(x, \theta)$ and we can determine the divergent beam transform from the x-ray data.

The formal adjoint of $\mathcal{D}_x$ can be found from the following result.

Proposition 5.2.

$$\int_{S^{n-1}} \mathcal{D}_a f(\theta) h(\theta) d\theta = \int_{\mathbb{R}^n} f(x) |x - a|^{1-n} h\left(\frac{x - a}{|x - a|}\right) dx \quad (5.2)$$

The formula is valid if f and h are nonnegative measurable functions, although the result may be infinite. The equation is valid and finite if either side is finite when f and h are replaced by $|f|$ and $|h|$.

Furthermore, if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{D}_a f \in L^1(S^{n-1})$.

Proof. Using polar coordinates centered at a we have

$$\begin{aligned} & \int_{\mathbb{R}^n} f(x) |x - a|^{1-n} h\left(\frac{x - a}{|x - a|}\right) dx \\ &= \int_{S^{n-1}} \int_0^\infty f(a + s\theta) |s\theta|^{1-n} h(\theta) s^{n-1} ds d\theta \\ &= \int_{S^{n-1}} \int_0^\infty f(a + s\theta) h(\theta) ds d\theta \\ &= \int_{S^{n-1}} \mathcal{D}_a f(\theta) h(\theta) d\theta. \end{aligned}$$

The statements about validity and finiteness are direct consequences of Fubini's theorem

The statement about the integrability of $\mathcal{D}_a f$ follows by taking $h \equiv 1$. ■

We can interpret the results of this proposition by the following inner product formula

$$\langle \mathcal{D}_a f(\theta), h(\theta) \rangle_{S^{n-1}} = \left\langle f(x), |x - a|^{1-n} h\left(\frac{x - a}{|x - a|}\right) \right\rangle_{\mathbb{R}^n}$$

and from this the following definition makes sense.

Definition 5.3. The formal adjoint $\mathcal{D}_a^\#$ of the divergent beam transform at the point $a \in \mathbb{R}^n$ is defined by

$$\mathcal{D}_a^\# h(x) = |x - a|^{1-n} h\left(\frac{x - a}{|x - a|}\right).$$

Most studies of the divergent beam transform are concerned with some form of the restricted transform. Historically, the mathematical development paralleled the practical applications of tomography. In practice one would not have x-ray sources at every point in space, nor even a useful approximation thereof. Therefore the development of the theory concentrated on various forms of the restricted divergent beam transform.

The simplest situation for the restricted divergent beam transform occurs when we restrict attention to functions with support inside a sphere in $\mathbb{R}^n$. Referring back to figure 5.2 we see that as the direction θ varies we obtain the densities of f along all the rays emanating from the point x . As shown in the upper ray of the figure, $\mathcal{D}f(x, \theta) = 0$ if the corresponding ray misses the support of f . By restricting x to lie on a sphere surrounding the support of f , we can find the total densities of f along any line in $\mathbb{R}^n$. This essentially gives the same information as the x-ray transform.

Therefore, the divergent beam transform restricted to a sphere can be inverted by using the inverse of the x-ray transform. To do this requires resorting the data from the divergent beam transform, placing all rays parallel to each other in one array. An algorithm to accomplish this is called a **rebinning algorithm**. However, a direct algorithm for inverting this transform is of interest. This is the content of theorem 5.4.

The inversion theorem for the divergent beam transform that we present now uses the Λ operator, introduced in definition 3.39, chapter 3, and the coefficient $\rho_{k,n}$ of the Riesz kernel, introduced in equation (3.55), of chapter 3.

Theorem 5.4. *If Ω is a compact subset of $\mathbb{R}^n$, if $f \in L^2(\Omega)$, and if $\mathfrak{S}$ is an $n - 1$ sphere of radius r , such that the convex hull of Ω is contained in the ball whose boundary is $\mathfrak{S}$, then for almost every $x \in \Omega$ we have*

$$f(x) = \frac{\rho_{1,n}}{2r} \Lambda \int_{a \in \mathfrak{S}} \left(\mathcal{D}_a f \left(\frac{x-a}{|a-x|} \right) + \mathcal{D}_a f \left(\frac{a-x}{|a-x|} \right) \right) \\ \times |a-x|^{-n} |\langle a, a-x \rangle| da$$

where da is Lebesgue measure on the sphere $\mathfrak{S}$.

Remark 5.5. The integral on the right-hand side of the formula is a function of x and it is that function of x that Λ operates on. This requires the knowledge of $\mathcal{D}_a f(\theta)$ for every $a \in \mathfrak{S}$ and every $\theta \in S^{n-1}$. For each source position a we can ignore directions whose rays do not intersect Ω ; however, some source points a may require more directions than other source points.

The proof of this theorem depends on the next result whose proof can be found in the appendix to this chapter.

Lemma 5.6. *Let $\mathfrak{S}$ be an $n - 1$ sphere of radius r , let x be a point in $\mathbb{R}^n$ which is inside $\mathfrak{S}$ and let f be an integrable function on S^{n-1} . Then*

$$\int_{S^{n-1}} f(\theta) d\theta = \frac{1}{r} \int_{\mathfrak{S}} f \left(\frac{a-x}{|a-x|} \right) |a-x|^{-n} |\langle a, a-x \rangle| da$$

where $d\theta$ is Lebesgue measure on the unit sphere S^{n-1} and da is Lebesgue measure on the sphere $\mathfrak{S}$.

Proof of theorem 5.4. Recall from chapter 3, that

$$\int_{\mathbb{R}^n} f(y) |y - x|^{1-n} dy = \frac{1}{\rho_{1,n}} R_1 * f(x) \quad (5.3)$$

where $R_1 * f$ is the Riesz potential (see definition 3.55).

If we take $h = 1$ in proposition 5.2, then we also get

$$\int_{\mathbb{R}^n} f(y) |y - x|^{1-n} dy = \int_{S^{n-1}} \mathcal{D}_x f(\theta) d\theta \quad (5.4)$$

The transformation $\theta \rightarrow -\theta$ is orthogonal, so by the invariance of the Haar measure on the sphere we have

$$\int_{S^{n-1}} \mathcal{D}_x f(-\theta) d\theta = \int_{S^{n-1}} \mathcal{D}_x f(\theta) d\theta \quad (5.5)$$

From equations (5.3)–(5.5) and lemma 5.6 we obtain

$$\begin{aligned} 2 \frac{1}{\rho_{1,n}} R_1 * f(x) &= \int_{S^{n-1}} \mathcal{D}_x f(\theta) d\theta + \int_{S^{n-1}} \mathcal{D}_x f(-\theta) d\theta \\ &= \frac{1}{r} \int_{\mathfrak{S}} \mathcal{D}_x f\left(\frac{a-x}{|a-x|}\right) |a-x|^{-n} |\langle a, a-x \rangle| da \\ &\quad + \frac{1}{r} \int_{\mathfrak{S}} \mathcal{D}_x f\left(\frac{x-a}{|a-x|}\right) |a-x|^{-n} |\langle a, a-x \rangle| da \end{aligned}$$

But it is easily checked that the divergent beam transform has the following symmetry property

$$\mathcal{D}_x f\left(\frac{a-x}{|a-x|}\right) + \mathcal{D}_x f\left(\frac{x-a}{|a-x|}\right) = \mathcal{D}_a f\left(\frac{a-x}{|a-x|}\right) + \mathcal{D}_a f\left(\frac{x-a}{|a-x|}\right)$$

Therefore,

$$\begin{aligned} \frac{1}{\rho_{1,n}} R_1 * f(x) & \\ &= \frac{1}{2r} \int_{\mathfrak{S}} \left(\mathcal{D}_a f\left(\frac{a-x}{|a-x|}\right) + \mathcal{D}_a f\left(\frac{x-a}{|a-x|}\right) \right) |a-x|^{-n} |\langle a, a-x \rangle| da \end{aligned} \quad (5.6)$$

Since f is a compactly supported L^2 function, then from a corollary to the Riesz multiplier theorem (chapter 3, corollary 3.76) we know that in this case we have $\Lambda(R_1 * f) = f$. Therefore, applying Λ to equation (5.6) gives the desired result. ■

Theorem 5.4 is from Leahy, Smith, and Solmon [383]. This reference [383] exists only as a preprint (never having been published). However, their results have been reproduced in Hamaker et al. [254], Smith [562], and Keinert [348].

It is possible to improve the situation. It is impractical to have x-ray sources at all or even many points on a sphere surrounding the object of interest. Therefore, we investigate the conditions under which the divergent beam transform restricted to a curve in $\mathbb{R}^3$ may be inverted. This is a practical situation and there are CT scanners that gather data from fan beam sources distributed on a curve.

For simplicity we restrict attention to $\mathbb{R}^3$.

Definition 5.7. A curve γ in $\mathbb{R}^3$ satisfies the *Kirillov condition* ([353]) if every plane in $\mathbb{R}^3$ intersects the curve. A curve γ in $\mathbb{R}^3$ satisfies the *Tuy condition* ([604]) relative to the set X , if every plane which intersects X also intersects γ transversally.

Tuy [604] proved that one could reconstruct a compactly supported integrable function f on $\mathbb{R}^3$ by using the divergent beam transform with sources restricted to a curve satisfying the Tuy condition relative to the support of f . Before stating his theorem we need to make the following observations.

- We can extend $\mathcal{D}f$ from $\mathbb{R}^3 \times S^{n-1}$ to a function on $\mathbb{R}^3 \times \mathbb{R}^3$ that is homogeneous of degree -1 in the second variable. This is done by defining, for $y \in \mathbb{R}^3$,

$$\mathcal{D}f(a, y) = \int_0^\infty f(a + sy) ds.$$

For Tuy's theorem we will be interested only in transforms of the form $\mathcal{D}f(a(t), y)$, where $a(t)$ is a smooth curve in $\mathbb{R}^3$. Then it makes sense to consider the Fourier transform of $\mathcal{D}f(a(t), y)$ in the y variable. We denote this Fourier transform by $(\mathcal{D}f)^\wedge(a(t), \xi)$.

It also makes sense to differentiate $(\mathcal{D}f)^\wedge(a(t), \xi)$ with respect to t and we denote this derivative by $(\partial_t(\mathcal{D}f)^\wedge)(a(t), \xi)$.

- The Tuy condition can be made more explicit. A typical plane intersecting X consists of all points y satisfying the equation $\langle y, \theta \rangle = \langle x, \theta \rangle$ for some $x \in X$. If this plane also intersects the curve $a(t)$, then there is at least one parameter value t such that $\langle a(t), \theta \rangle = \langle x, \theta \rangle$. The transversality condition means that $\langle a'(t), \theta \rangle \neq 0$. Hence, if the curve satisfies the Tuy condition, then we can define a function $\tau : X \times S^3 \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \langle a(\tau(x, \theta)), \theta \rangle &= \langle x, \theta \rangle \\ \langle a'(\tau(x, \theta)), \theta \rangle &\neq 0 \end{aligned}$$

We state a version of Tuy's theorem [604] which is adapted from Natterer [444] (compare, Natterer and Wübbeling [446]). The reader can refer to these works for the proof.

Theorem 5.8 (Tuy). *Let f be a C^3 function supported on the unit ball $B(1)$ of $\mathbb{R}^3$ and let $a(t)$ be a curve in $\mathbb{R}^3$ satisfying the Tuy condition relative to the ball. Then, for all*

$x \in B(1)$ we have

$$f(x) = -(2\pi)^{-\frac{3}{2}} i \int_{S^2} \langle a'(\tau(x, \theta)), \theta \rangle^{-1} (\partial_t (\mathcal{D}f)^\wedge)(a(\tau(x, \theta)), \xi) d\theta.$$

Grangeat's method [229] gives an exact cone beam reconstruction for sources on a curve with the property that every plane hitting the support of f contains at least one source. This condition is somewhat more general than Tuy's condition. Grangeat's method can also be used for a filtered backprojection type of reconstruction from cone beam data. Also of interest are Palamodov [461], which contains Grangeat's method as a limiting case, Gel'fand and Goncharov [191], and Clack and Defrise [97], who unify and generalize the cone beam formulas of Tuy and Grangeat.

According to Natterer and Wübbeling [446] the FDK approximate formula is the most widely used formula for cone beam reconstruction with a circle as the source curve (Feldkamp, Davis, and Kress [167]). Although this method is useful, it is completely heuristic. Interested readers are referred to Natterer and Wübbeling [446] for details.

There are some interesting uniqueness and nonuniqueness results for the divergent beam transform.

Hamaker, Smith, and Solmon [254] proved the following uniqueness theorem for the divergent beam transform. The reader is referred to [254] for the proof.

Theorem 5.9. *If Ω is a compact subset of $\mathbb{R}^n$ and if A is an infinite subset of $\mathbb{R}^n$ bounded away from the convex hull of Ω , then for any $f \in L_0^1(\Omega)$ with $\mathcal{D}_a f = 0$ for each $a \in A$, we must have $f = 0$.*

Theorems 5.4 and 5.9 require the knowledge of $\mathcal{D}_a f(\theta)$ for all $\theta \in S^1$ for each $a \in A$. It is possible to obtain a uniqueness theorem in the situation where the beam is "coned down." This means that x-rays are measured only in certain directions emanating from a particular source point a . More precisely, let Ω be a compact subset of $\mathbb{R}^n$ and let C be an open cone with vertex 0. Then $a + C$ is an open cone with vertex a . Each half-line l in $a + C$ issuing from a represents the path of an x-ray in the coned down beam with source at a . If A is a subset of $\mathbb{R}^n$, then $A + C$ represents the path of all x-rays in the coned down beams emanating from points on A . We define the **measured region** of Ω to be the set Ω_m defined by $\Omega_m = \Omega \cap (A + C)$. Therefore, a point is in the measured region if and only if it is in Ω and has the form $a + t\theta$ for some $a \in A$, some $t > 0$ and some $\theta \in C \cap S^{n-1}$.

Hamaker et al. [254] have a simple example which shows that the uniqueness theorem does not hold in the coned down case for infinite sets of sources. However, the following theorem adapted from [254] is valid.

Theorem 5.10. *Let Ω be a bounded open set in $\mathbb{R}^n$. Also let $f \in L_0^1(\Omega)$, let C be an open cone with vertex 0, and let A be a rectifiable curve outside Ω such that for each half-line l in C , there is a point $a \in A$ such that $a + l$ misses Ω . Then if for each $a \in A$, $\mathcal{D}_a f(\theta) = 0$ for all $\theta \in C \cap S^{n-1}$, then $f = 0$ almost everywhere on the measured region Ω_m .*

Here is a sketch of the proof. Define $\mathcal{D}_a^k f(\theta) = \int_0^\infty t^k f(a + t\theta) dt$. It is possible to prove by induction that for integers $k \geq 0$ and for all $a \in A$:

$$\mathcal{D}_a^k f(\theta) = 0 \text{ for almost all } \theta \in C \cap S^{n-1}.$$

The hypothesis provides the initial step of the induction. The induction proof is rather nasty and requires the hypothesis that there is a point $a \in A$ such that $a + l$ misses Ω (see [254] for details).

Now fix $a \in A$ and fix θ such that $\mathcal{D}_a^k f(\theta) = 0$ for all $k \in \mathbb{Z}^+$. Define $\varphi(t) = f(a + t\theta)$. Since $f \in L_0^1(\Omega)$, then $\varphi \in L_0^1(\mathbb{R})$ and $\varphi(t) = 0$ for $t < 0$. Hence

$$\int_{-\infty}^{\infty} t^k \varphi(t) dt = \mathcal{D}_a^k f(\theta) = 0.$$

Letting $\check{\varphi}$ denote the inverse Fourier transform of φ , we can therefore apply the derivative theorem for the Fourier transform to get

$$0 = \int_{-\infty}^{\infty} t^k \varphi(t) dt = c \cdot \frac{d^k}{d\tau^k} \check{\varphi}(0).$$

By the Paley–Wiener theorem, the compactly supported L^1 function $\check{\varphi}$ has a Fourier transform that is an entire analytic function. But all the derivatives of $\check{\varphi}$ are zero at the origin, so $\check{\varphi}$, and therefore also φ , is zero almost everywhere. By the definition of φ we therefore obtain $f(a + t\theta) = 0$ for almost all t .

Since this result is true for any a and θ in the set where $\alpha \in A$ and $\theta \in C \cap S^{n-1}$, then $f(a + t\theta) = 0$ for almost all such a and θ and almost all $t > 0$. The resulting set of $a + t\theta$ forms the measured region, so $f = 0$ almost everywhere on Ω_m . ■

Remark 5.11. The authors actually showed that A can be a dense subset of a rectifiable curve. The proof is almost identical.

Remark 5.12. Approximate inversion formulas for the divergent beam transform may be found in Leahy, Smith, and Solmon [383] and Smith [562].

In their paper [174], Finch and Solmon give a range theorem and consistency conditions for the divergent beam transform with sources on a sphere. They also show that the divergent beam transform is a smoothing operator in the same sense that we showed that the k -plane transform is a smoothing operator: the k -plane transform of a function with j derivatives has $j + \frac{k}{2}$ derivatives, in the Sobolev space sense (compare, chapter 3, remark 3.100).

Because Sobolev spaces are required for the statement of the Finch and Solmon theorem, we give a brief summary of the relevant definitions. A knowledge of spherical harmonics is also necessary to understand one of the definitions.

Definition 5.13. If s is a nonnegative real number then the Sobolev space $H^s(\mathbb{R}^n)$ of order s on $\mathbb{R}^n$ is defined to be the set of tempered distributions u whose Fourier

transforms $\widehat{u}$ are locally square integrable and for which the following norm is finite

$$\|u\|_{H^s(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 (1 + |\xi|^2)^s ds \right]^{\frac{1}{2}}$$

Then $H^s(\mathbb{R}^n)$ is a Hilbert space under this norm.

If Ω is an open subset of $\mathbb{R}^n$, then the Sobolev space $H_0^s(\Omega)$ consists of all $u \in H^s(\mathbb{R}^n)$ with support in $\overline{\Omega}$.

Sobolev spaces can be defined for compact differentiable manifolds. For manifolds of the form $S^k \times S^l$ where S^k and S^l are spheres of dimension k and l , respectively, there is the following explicit method of defining the Sobolev space $H^{a,b}(S^k \times S^l)$. Let $\{Y_{p,q}^k\}$ be an orthonormal basis of the spherical harmonics on the sphere S^k with eigenvalues λ_p^k . Similar notation is used for the other sphere. If $u \in L^2(S^k \times S^l)$, then we let $C_{p,q,r,s}(u)$ denote the coefficients of the expansion of u relative to the orthonormal basis of $L^2(S^k \times S^l)$ determined by $\{Y_{p,q}^k\}$ and $\{Y_{r,s}^l\}$: this means that

$$u = \sum C_{p,q,r,s}(u) Y_{p,q}^k Y_{r,s}^l$$

The Sobolev space $H^{a,b}(S^k \times S^l)$ is defined to be the set of such u for which the following norm is finite

$$\|u\|_{H^{a,b}(S^k \times S^l)} = \left[\sum C_{p,q,r,s}^2(u) (1 + \lambda_p^k)^a (1 + \lambda_r^l)^b \right]^{\frac{1}{2}}$$

For the reader who is not familiar with Sobolev spaces, Natterer [444], section VII.4, has an excellent quick introduction to the basic definitions and properties of Sobolev spaces on $\mathbb{R}^n$. Triebel [603] is a complete reference to the Sobolev space theory (also, compare, Finch [168]).

We now state the range theorem of Finch and Solmon [174], which we mentioned earlier. We refer the reader to [174] for all the proofs and further details.

Theorem 5.14 (Finch and Solmon [174]). *Let Ω be an open, relatively compact, convex subset of $\mathbb{R}^n$, and let S_r be a sphere of radius r such that the corresponding open ball B_r contains the closure of Ω . We consider the transform $\mathcal{D}$ restricted to $L_0^2(\Omega)$. Then the range of $\mathcal{D}$ is contained in the Sobolev space $H^{\frac{1}{2},0}(S_r \times S^{n-1})$ and $\mathcal{D} : L_0^2(\Omega) \rightarrow H^{\frac{1}{2},0}(S_r \times S^{n-1})$ is a continuous operator with continuous inverse.*

The range of $\mathcal{D}$ is characterized by the following consistency conditions: a measurable function g is in the range of $\mathcal{D}$ if and only if

1. $g \in H^{\frac{1}{2},0}(S_r \times S^{n-1})$,
2. $g(a, \theta) = 0$ if the line through a with direction θ misses Ω ,
3. $|(a, \theta)|g(a, \theta) = |(b, \theta)|g(b, \theta)$ if the orthogonal projections of a and b on $\theta^\perp$ are the same, and

4. for each integer $m \geq 0$, the function that maps $\theta \in S^{n-1}$ and $\xi \in \theta^\perp$ to the integral

$$\int_{S_r} \langle a, \xi \rangle^m |\langle a, \theta \rangle| g(a, \theta) d\theta$$

is independent of θ and defines a function $q_m(\xi)$, which is a homogeneous polynomial of degree $\leq m$.

Note the similarity of this theorem to the range theorems 4.12 and 4.24 in chapter 4.

The next set of results involves the concept of the generalized inverse of an operator. If $A : H \rightarrow K$ is an operator between two Hilbert spaces, then the generalized inverse maps an element $g \in K$ to an element $f \in H$ such that f is the minimizer of $\|Af - g\|$. The generalized inverse is clearly equal to A^{-1} if A is invertible and it is a useful substitute for the inverse if A is not invertible. The generalized inverse exists in a wide variety of situations but is not always continuous. A Hilbert space operator is said to be stable if the generalized inverse exists and is continuous. This occurs if and only if the operator has a closed range. For more details the reader may refer to Natterer [444] for a brief but useful introduction to generalized inverses.

We use the notation $L^2(X, \mu)$ to denote the Hilbert space of square integrable functions relative to the measure μ .

In general, Radon transforms do not have continuous inverses. But in certain cases, for example when restricted to functions supported in a compact set, they do have continuous inverses in a Sobolev space topology.

The continuity of the inverse of $\mathcal{L} : H_0^s(\Omega) \rightarrow H^{s+\frac{1}{2},0}(S_r \times S^{n-1})$ is referred to as “stability of $\mathcal{L}$ in the scale of the Sobolev spaces.” This means that a small change in $\mathcal{L}f$ yields a correspondingly small change in f itself. Therefore, part of the previous result can be interpreted as a stability theorem for $\mathcal{L}$ for the case in which $s = 0$. Finch and Solmon remark that this can be extended to arbitrary positive s .

Theorem 5.14 applied to the divergent beam transform on a set of positive measure in $\mathbb{R}^n$. If we restrict the divergent beam transform to have sources on a curve, then there may or may not be Sobolev stability. Let $\mathcal{D}$ denote the divergent beam transform with sources on a curve C in $\mathbb{R}^3$. Finch [168] shows that there is, in general, no Sobolev stability for $\mathcal{D}$. In fact, he shows that if C is a curve in $\mathbb{R}^3$ such that there exists a plane that passes through Ω but misses C , then if the inverse of $\mathcal{D}$ exists, then it is not continuous. This is a consequence of his theorem 4.9 [168], which gives an estimate showing that even the generalized inverse is not continuous under these circumstances. However, Finch proceeds to show that if the convex hull of C contains a sphere and satisfies Tuy’s condition, then for $s \geq 0$ the inverse of $\mathcal{D} : H_0^s(\Omega) \rightarrow H^{s+\frac{1}{2},0}(C \times S^2)$ is continuous (theorem 4.30 [168]). In this paper, Finch also shows that for $f \in C_0^2(\Omega)$, if $\mathcal{D}f(x, \theta)$ is known on a connected curve in $\mathbb{R}^3 \setminus \overline{\Omega}$, then it can be determined in an explicit fashion for x in the convex hull of C (theorem 3.7). He goes on to prove that if there is a nonempty open cone Γ such that for all $\theta \in \Gamma \cap S^2$, the orthogonal projection

of the support of f onto $\theta^\perp$ is contained in the projection of the convex hull of C onto $\theta^\perp$, then f can be determined from $\mathcal{D}f$ with sources restricted to the curve C . This is a consequence of Finch's theorem 3.11 [168], which shows that if the x-ray transform of a compactly supported, square integrable function f is known for all $\theta \in \Gamma \cap S^2$, where Γ is a nonempty open cone, then the Fourier transform of f , and hence f itself, can be determined. The reader is referred to Finch [168] for all proofs and details.

Stability results exist for the divergent beam transform with a finite number of sources.

Finch and Solmon [172] give conditions for stability of the divergent beam transform on $\mathbb{R}^2$ with a finite number of sources. Let Ω be an open, convex, bounded subset of $\mathbb{R}^2$. Let $a \in \mathbb{R}^2 \setminus \overline{\Omega}$. Finch and Solmon define a weight function δ_a on the sphere S^1 in such a way that it is easy to show that the map $\mathcal{D}_a : L^2(\Omega) \rightarrow L^2(S^1, \delta_a(\theta)d\theta)$ is bounded and has closed range.

Therefore, the operator $\mathcal{D}_a$ is stable for one source. Does stability persist if there is more than one source? The answer is yes if there are finitely many sources and if a certain condition is satisfied. Let $A = \{a_1, \dots, a_m\} \subset \mathbb{R}^2$ be a finite set of source points none of which is contained in $\overline{\Omega}$. In this case Finch and Solmon define the operator

$$\mathcal{D}_A : L^2(\Omega) \rightarrow \bigoplus_{a \in A} L^2(S^1, \delta_a(\theta)d\theta)$$

in the obvious way and they introduce the following condition:

- *Condition E.* For each pair of sources a_i, a_j in A , the line segment connecting a_i to a_j does not intersect $\overline{\Omega}$.

Finch and Solmon prove that condition *E* is sufficient for the stability of $\mathcal{D}_A$. They provide an example involving only a single pair of sources $\{a_1, a_2\}$, for which the line determined by these sources intersects $\overline{\Omega}$ in only a single point and for which $\mathcal{D}_A$ is not stable. They also give an example to show that condition *E* is not necessary for the stability of $\mathcal{D}_A$. Finally, they give a characterization of the closure of the range of $\mathcal{D}_A$ under condition *E* and the further condition that no three sources lie on the same line. We refer the reader to [172] for proofs and further details.

5.3 Attenuated and Exponential Radon Transforms

There are other tomographic methods in diagnostic radiology besides CT scanning. For example, there is *single photon emission computerized tomography (SPECT)* and *positron emission tomography (PET)*. These types of tomography are modeled by certain Radon transforms. PET tomography reduces to the ordinary Radon transform, but SPECT tomography uses an instance of a generalized Radon transform. We will study generalized Radon transforms in detail in section 5.4.

CT reconstructs an object from photon counts of x-rays produced by a source external to the object. These x-rays traverse the entire region in which the object is located. However, in both SPECT and PET, the patient is injected with a radioactive substance (radionuclide) in the area of diagnostic interest. The extent to which various normal and

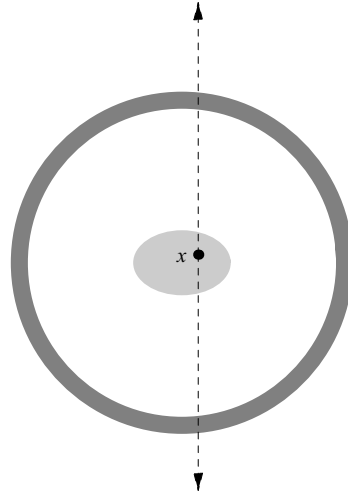


Figure 5.3. Model of positron emission tomography (PET).

pathological tissue absorb the radionuclide provides important diagnostic information. Therefore, it is desired to reconstruct the distribution of the radionuclide rather than the density of the tissue in which this material is located.

In PET the radionuclide emits positrons. Each positron almost instantly annihilates a neighboring electron. This results in the formation of two gamma rays that issue in opposite directions from the position of the annihilated electron. A pair of gamma-ray detectors is placed around the body, one hundred and eighty degrees from each other. These detectors can recognize the simultaneous arrival of two oppositely oriented gamma rays.

Figure 5.3 illustrates the idea of PET scanning. The large circle represents the boundary of a body. However, no details of the inner structure are shown. The light-gray ellipse represents the injected radionuclide. At the point x , an electron-positron annihilation has occurred, resulting in two oppositely oriented gamma rays. The gamma rays are represented by the dotted lines. We have not shown the detectors, but in practice this event would be detected because the gamma rays arrive at the opposite detectors essentially simultaneously.

This configuration provides an account of all the positron emission on the line between the two detectors. When many such events are detected, the density of radioactive substance along the line of travel can be determined. By using an array of such detectors, information about positron events on each line through the region of interest can be determined. This suggests that the distribution of the radionuclide can be determined by using the Radon transform. This is essentially correct, except that the effect of the density of the tissue has to be factored out. The details of this procedure may be found in Natterer and Wübbeling [446]. However, there are no new mathematical ideas involved in this procedure. In addition to reference [446], the reader can find some interesting details on the physics of positron emission tomography in Deans [124]. Both of these books also contain many bibliographic references to PET scanning.

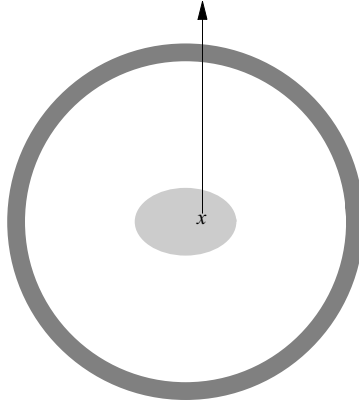


Figure 5.4. Model of single photon emission tomography (SPECT).

In SPECT, radioactive material, whose density is denoted by f , is injected into the patient. The radioactive material emits photons which travel along straight lines in all directions. Thereby beams of photons are created, emanating in all directions from any point where f is nonzero. Each such beam will be attenuated by surrounding tissue until it hits a photon counter (detector). The density of the surrounding tissue is denoted by the function μ . The observable data are the photon counts at each detector. As we know from the Lambert-Beer law, the photon counts at the detector obey an exponential law. Figure 5.4 illustrates the idea of SPECT scanning. It is similar to figure 5.3 except that the radionuclide emits photons and each photon travels along a ray in a specific direction. We have illustrated the path of one of the photons emanating from a particular point in the radionuclide. However, photons are being emanated in all directions from this point, and, indeed, from every point in the radionuclide. If an array of photon detectors is placed around the body, then we can determine the total number of photons that travel along a ray in any direction. However, we cannot determine the origin of any individual ray.

The following definition, in which $\mathcal{D}$ is the divergent beam transform from definition 5.1, models this situation.

Definition 5.15 (Attenuated x-ray transform). The *attenuated x-ray transform* is defined by

$$P_{\mu}f(\theta, x) = \int_{-\infty}^{\infty} f(x + t\theta) e^{-\mathcal{D}\mu(x+t\theta, \theta)} dt$$

The function μ is called the attenuation function and the function f is called the activity function of the attenuated x-ray transform. In SPECT the attenuation function represents the attenuation coefficients of the material in which the photon source is embedded and the activity function represents the photon source.

To see how definition 5.15 relates to the process of single photon emission computerized tomography, let us begin by analyzing what happens if there is a small amount of radioactive material at the point x of $\mathbb{R}^2$, which sits in a medium of density μ and

emits f_0 photons uniformly in all directions. We are particularly interested in what happens in a specific direction θ . Assume that there is a detector that can count the photons emitted in this direction. We may assume that the detector is located at infinity. According to the Lambert-Beer law, section 1.9, the number of photons counted at the detector will be $f_0 e^{-\int_L \mu(s) ds}$, where L is the ray in the direction θ and where s is the arc length parameter along that line. However, we can parametrize that ray by $x + s\theta$, s ranging from 0 to ∞ , so the number of photons detected will be

$$f_0 \exp \left(- \int_0^{\infty} \mu(x + s\theta) ds \right) = f_0 e^{-\mathcal{D}\mu(x, \theta)}$$

where $\mathcal{D}$ is the divergent beam transform, definition 5.1. If we now take an infinitesimal segment dt located at the point $x + t\theta$ of the line through x in the direction θ , the number of photons generated on that segment and counted at the detector will be approximately $f(x + t\theta) e^{-\mathcal{D}\mu(x + t\theta, \theta)} dt$. Adding up all these contributions gives the integral defining $P_\mu f(\theta, x)$.

The attenuated Radon transform can also be defined.

Definition 5.16. The *attenuated Radon transform* is defined by

$$R_\mu f(\theta, s) = \int_{\langle x, \theta \rangle = s} f(x) e^{-\mathcal{D}\mu(x, \theta^\perp)} dx$$

The function μ is called the attenuation function and the function f is called the activity function of the attenuated Radon transform.

Remark 5.17. Some authors use the weight $e^{\mathcal{D}\mu(x, \theta^\perp)}$. This is of no mathematical consequence.

In dimension two, which is the main case of interest in this section, the two transforms are essentially the same. The attenuated Radon transform is an instance of the generalized Radon transform (see section 5.4). We do not invert the attenuated x-ray transform now because that is done in section 5.4. However, the history of the attenuated Radon transform and attempts to invert it are interesting and we present a brief survey of the subject now.

According to Tretiak [600], SPECT is the earliest form of computerized tomography going back to Kuhl and Edwards [366]. The computer used by Kuhl and Edwards was not a digital computer, but rather an optical integrator. They also introduced the idea of transverse section scanning and the use of backprojections to approximate the distribution.

As far as the claim of primacy in CT, however, Korenblyum, Tetelbaum, and Tyutin [357] published a paper in 1958 outlining the mathematical procedure for fan beam scanning and describing an experiment in which a 100×100 pixel image was reconstructed from x-ray projections. This apparently was never clinically implemented and, as of 1983, no further evidence of publications on CT were found in the Russian literature (compare, section 1.10.1).

There was, and still is, a parallel interest in SPECT from both the mathematical and medical sides. The earliest mathematical contributions seem to be those of Tretiak and Delaney [601] in 1978, Bellini et al. [40] in 1979, Natterer [435] in 1979, Tretiak and Metz [602] in 1980, and Natterer [437] in 1981.

There were many attempts to show that the attenuated Radon transform was invertible and to produce an explicit inversion formula. The best result before 1998 was from Finch [169]. Let Ω be a bounded open set in $\mathbb{R}^2$ and let μ be a C^∞ attenuation function for the attenuated Radon transform on $\mathbb{R}^2$. Then Finch was able to show that R_μ is injective on functions supported in Ω provided that $\|\mu\|_{L^\infty} \text{diam}(\Omega) < 5.37$. His proof did not apply to domains where this product is larger. However, Finch gave a brief and clever argument that the restricted invertibility result in $\mathbb{R}^2$ leads to the global invertibility of the attenuated Radon transform on any $\mathbb{R}^n$ with $n \geq 3$. The proof of this fact depends on the local invertibility of the attenuated Radon transform on $\mathbb{R}^2$. Local invertibility is known for generalized Radon transforms on $\mathbb{R}^2$, as shown by the result of Markoe and Quinto, theorem 5.35, which is proved later. With this observation, Finch remarks that, in fact, his proof of global invertibility applies to generalized k -plane transforms on $\mathbb{R}^n$ provided that $k \leq n - 2$.

It was still not known, however, whether the attenuated Radon transform is invertible on $\mathbb{R}^2$. This question remained open until 1998 when Arbuzov, Bukhgeim, and Kazantsev [24] were able to prove that the attenuated Radon transform is globally invertible. They did not produce an explicit inversion formula, although Finch [170] showed how it is possible to do this with their methods. Novikov [453] seems to be the first one to have produced an explicit inversion formula for the attenuated Radon transform. His results were made known in 2000 but were not formally published until 2002. In 2001 Natterer [443], with the knowledge of Novikov's results, presented an inversion formula similar to Novikov's but with a simpler proof. In 2003 Boman and Strömberg developed an inversion formula for a generalized Radon transform related to, but more general than the attenuated Radon transform. We will describe the ideas of Boman and Strömberg, but in a later section (see theorem 5.56 in section 5.4). An inversion formula for the attenuated Radon transform is a consequence of the Boman–Strömberg results (see theorem 5.64).

5.3.1 The Exponential Radon Transform

It was discovered very early that the problem of inverting the attenuated Radon transform was significantly more difficult than inverting the Radon transform. However, if the attenuation coefficient μ is constant, then it is relatively easy to invert the attenuated Radon transform. This was done by the early researchers Tretiak and Delaney [601], Bellini et al. [40], Natterer [435], Tretiak and Metz [602], and Natterer [437].

The technique in the case of known, constant attenuation is to reduce the attenuated Radon transform to a simpler transform called the exponential Radon transform. Actually it is enough to assume that the attenuation μ is constant on a convex set containing the activity distribution f . This clever idea is from Tretiak and Delaney [601]. For

simplicity we restrict attention to a very simple convex set, the unit disk in $\mathbb{R}^2$. Later we will comment on the more general situation.

Let the attenuation function μ be equal to the real valued constant μ_0 on $B(1)$ and let it be equal to zero elsewhere. Then for any $x \in B(1)$ we have

$$\mathcal{D}\mu(x, \theta^\perp) = \int_0^\infty \mu(x + s\theta^\perp) ds = \mu_0 d(x, \theta)$$

where

$$d(x, \theta) = \left| \sqrt{1 - (x \cdot \theta)^2} - x \cdot \theta^\perp \right|$$

is the distance from the point x to the boundary of the disk in the direction $\theta^\perp$. Hence, $R_\mu f(\theta, p) = \int_{\langle x, \theta \rangle = p} e^{-\mu_0 d(x, \theta)} f(x) dx$. If we now introduce coordinates on the line $\langle x, \theta \rangle = p$ using the basis $\{\theta, \theta^\perp\}$, we see that $x = p\theta + t\theta^\perp$ and $d(x, \theta) = \sqrt{1 - p^2} - t$ and, hence,

$$\begin{aligned} R_\mu f(\theta, p) &= \int_{\mathbb{R}} e^{-\mu_0(\sqrt{1-p^2}-t)} f(p\theta + t\theta^\perp) dt \\ &= e^{-\mu_0\sqrt{1-p^2}} \int_{\mathbb{R}} e^{\mu_0 t} f(p\theta + t\theta^\perp) dt \end{aligned} \quad (5.7)$$

Since the attenuation μ_0 is known, it follows from this equation that the attenuated Radon transform with constant attenuation is defined completely in terms of the integral operator on the right-hand side.

Basically the same reasoning applies if the disk is replaced by an arbitrary compact, convex set on which μ_0 is constant. Treiaki and Delaney [601] have the details, but Markoe [423] or Natterer [444] are more accessible.

We therefore make the following definition.

Definition 5.18. Let μ be a real number. The *exponential Radon transform* T_μ is defined by

$$T_\mu f(\theta, p) = \int_{\langle x, \theta \rangle = p} f(x) e^{\mu \langle x, \theta^\perp \rangle} dx$$

for $(\theta, p) \in S^{n-1} \times \mathbb{R}$.

In the case $n = 2$ we can write this as

$$T_\mu f(\theta, p) = \int_{-\infty}^{\infty} f(p\theta + t\theta^\perp) e^{\mu t} dt$$

for (θ, p) in the cylinder $S^1 \times \mathbb{R}$.

Remark 5.19. If $\mu \equiv 0$, then the exponential Radon transform is the same as the usual Radon transform.

Remark 5.20. It is clear from this definition and equation (5.7) that in the case of constant attenuation, the inversion of the attenuated Radon transform is equivalent to the inversion of the exponential Radon transform.

We can also define an exponential x-ray transform.

Definition 5.21. If μ is a real number, then the *exponential x-ray transform* X_μ is defined by

$$X_\mu f(\theta, x) = \int_{-\infty}^{\infty} f(x + t\theta) e^{\mu t} dt$$

Remark 5.22. In dimension $n = 2$ the exponential Radon transform T_μ with constant weight μ is essentially the same as the exponential x-ray transform X_μ .

Tretiak and Delaney [601] were the first to have the clever idea, described before definition 5.18, that the attenuated Radon transform with constant attenuation in a convex neighborhood of the activity distribution can be reduced to the exponential Radon transform. Unfortunately the journal in which [601] appeared is fairly obscure, which may be why Natterer [444] cited the present author who independently published this result in Markoe [423]. However credit for this idea is due to Tretiak and Delaney, although Markoe [423] is more accessible.

The ability to deal with the situation in which the attenuation is constant on a convex set is very fruitful. Most anatomical regions do not have constant attenuation. However, in many important diagnostic situations, the attenuation is essentially constant in a neighborhood of the radionuclide. For example, if a radionuclide with unknown distribution f is injected into the liver, then the attenuation is not constant because there are nearby organs, for example, kidneys, colon, spine, etc., with different attenuations. However, it is likely that nearby the support of f , that is, in the liver, the attenuation is constant or nearly so. In such a case the exponential Radon transform can be used to recover the distribution of the radionuclide. Even so, situations do exist where the assumption of constant attenuation is not valid, and for this reason it is important to have an inversion formula for the attenuated Radon transform. We describe such a formula in section 5.4.

As far as the inversion of the exponential Radon transform goes there have been many contributions from 1978 to the present time. We now describe some of the main results. The reader is referred to the cited papers for details.

The first inversions of the exponential Radon transform were given by Tretiak and Delaney [601] and Bellini et al. [40]. At about that time Natterer [435] gave an approximate inversion formula and Tretiak and Metz [602] improved on the results of Tretiak and Delaney. Inversion formulas and injectivity results for the exponential Radon transform were also given by Quinto [501], Hertle [301], Markoe [423], Finch and Hertle [171], Hazou [259], Hazou and Solmon [260–262], Solmon [573], Hawkins, Yang, and Lechner [258], Kuchment and Shneiberg [365], and Palamodov [463].

Kuchment and Shneiberg [365] and Markoe [423] showed how to invert the exponential x-ray transform by recovering its Fourier transform. As noted above, this implies an inversion formula for the attenuated x-ray transform with attenuation constant in a convex neighborhood of the activity distribution. The method is to use Natterer's slice-projection theorem for the exponential Radon transform, theorem 2.1 in [435]. If we are given a compactly supported L^2 function on $\mathbb{R}^n$, then the Paley–Wiener theorem implies that the Fourier transform of $T_\mu f$ extends to a holomorphic function on $\mathbb{C}^n$. Natterer's slice-projection theorem then gives values of the Fourier transform on a certain submanifold in complex Euclidean space based on the values of the Fourier transform of $T_\mu f$. There are enough points contained in this manifold to apply Cauchy's integral formula to recover the Fourier transform of f at any point of $\mathbb{R}^n$. Then Fourier inversion gives the function f . Clearly the inversion formula gives an injectivity result. Markoe also proves a limited angle theorem.

Hazou and Solmon [260] proved a filtered backprojection theorem and an inversion theorem for the exponential x-ray transform. In chapter 2, section 2.7, we proved the filtered backprojection theorem which stated that $R^\#(w) * f = R^\#(w * Rf)$ for suitable functions f and w . We explained how this formula could be used to approximately invert the Radon transform. A similar formula holds for the x-ray transform. Hazou and Solmon derived the analogous formula for the exponential x-ray transform: $X_{-\mu}^\#(w) * f = X_{-\mu}^\#(w * X_\mu f)$. As a consequence they were able to derive an inversion formula similar to the inversion formula $\frac{1}{(2\pi)^k |G_{k,n-1}|} \Lambda^\alpha P^\# \Lambda^{k-\alpha} P f = f$ for the k -plane transform that we derived in chapter 3: the case where $k = 1$ and $\alpha = 0$ reads $\frac{1}{2\pi |G_{1,n-1}|} X^\# \Lambda X f = f$. The inversion formula of Hazou and Solmon for the exponential x-ray transform is $\frac{1}{2\pi |G_{1,n-1}|} X_{-\mu}^\# \Lambda_\mu X_\mu f = f$, where Λ_μ is an appropriate variation of the lambda operator defined in chapter 2. These results generalize those of Tretiak and Metz [602] who did the case $n = 2$. In their sequel [262] Hazou and Solmon tested the approximate inversion formula with numerical data.

In studying Radon transforms one is interested in injectivity theorems, limited-angle theorems, and support or hole theorems. An injectivity theorem merely states that the transform is injective, or has a trivial null space, on some linear space of functions. The limited-angle theorem states that a function f is determined by the values of $\mathcal{R}_\mu f(\theta, p)$, where θ ranges through any open subset of the sphere. The support or hole theorem states that f is supported on the ball $|x| < r$ if and only if, for each θ , $\mathcal{R}_\mu f(\theta, p)$ is supported on the set where $|p| < r$.

Hertle [301] introduced a more general exponential Radon transform in which the weight is allowed to depend on the direction. It is defined as follows:

Definition 5.23. Let ν be a vector valued measurable function defined on the unit sphere S^{n-1} . Define the **Hertle exponential Radon transform** R_ν with weight ν by

$$R_\nu f(\theta, p) = \int_{\langle x, \theta \rangle = p} f(x) \exp(x \cdot \nu(\theta)) dx.$$

Remark 5.24. In dimension $n = 2$ we can express $x = p\theta + t\theta^\perp$ in the local coordinate system $\{\theta, \theta^\perp\}$. We can also express $\nu(\theta) = \nu_1(\theta)\theta + \nu_2(\theta)\theta^\perp$. The Hertle

exponential Radon transform then takes the form

$$R_v f(\theta, p) = e^{p v_1(\theta)} \int_{\mathbb{R}} e^{v_2(\theta)t} f(p\theta + t\theta^\perp) dt.$$

In the special case that $v_1(\theta) = 0$ and $v_2(\theta) = \mu$ is a real constant, then the Hertle exponential Radon transform takes the form $R_v f(\theta, p) = \int_{\mathbb{R}} e^{\mu t} f(p\theta + t\theta^\perp) dt$ and is the same as the standard exponential Radon transform.

Hertle [301], in dimension $n = 2$, and Finch and Hertle [171], for arbitrary n , prove injectivity, a limited angle theorem and a support or hole theorem for the Hertle exponential Radon transform. Hertle [301] proves these results in the case that $n = 2$ and f is a compactly supported distribution (for the hole theorem he requires the vector field to be C^1 and f to be Lipschitz continuous). Finch and Hertle [171] extend these results to arbitrary n . The injectivity theorem and the limited-angle theorem are proved for $f \in L^1(\mathbb{R}^n)$ such that f is exponentially decreasing in a sense defined in the paper. The class of such f include compactly supported L^1 functions. The hole theorem is proved for compactly supported L^2 functions. They also have a Fourier inversion procedure in the case $n = 2$ which generalizes and simplifies Markoe's formula [423] for the Fourier inversion of the exponential x-ray transform. See also Shneĭberg [558] and Shneĭberg et al. [559] for inversion formulas for the exponential Radon transform and exponential x-ray transform in $\mathbb{R}^n$. Kuchment and Shneĭberg [365] develop an inversion formula for the Hertle exponential Radon transform based on the Cauchy theorem in several complex variables. Palamodov [463] describes an inversion formula for the Hertle exponential Radon transform on $\mathbb{R}^3$, which uses only data on a closed curve on the sphere.

In section 5.4 we introduce the idea of a generalized Radon transform by allowing one to integrate functions over hyperplanes with a different measure for each hyperplane. There are many interesting examples of such generalized Radon transforms, including the classical Radon transform, the attenuated Radon transform and the exponential Radon transform. It is interesting that the filtered backprojection idea is very limited in relation to generalized Radon transforms. In section 5.4, we describe a result of Hazou and Solmon which implies that a generalized Radon transform that obeys the filtered backprojection theorem must be of a very restricted type: it must be a Hertle exponential Radon transform. Therefore, for other species of generalized Radon transforms, we must look elsewhere than filtered backprojection for inversion formulas.

All the inversion theorems for attenuated or exponential Radon transforms assume that the weight function μ is known and try to determine the activity function f from $\mathcal{R}_\mu f$. A more ambitious project is to try to identify both an unknown activity function together with an unknown attenuation function. This is known as the **identification problem** for the attenuated Radon transform.

The identification problem cannot be solved in general. Natterer [437] has a simple counterexample for the case of the exponential Radon transform. However, it is possible to obtain some information about the identification problem if we restrict the input functions suitably. Natterer [437, 438] showed that given the attenuated Radon transform $R_\mu f$ for a generalized function f on $\mathbb{R}^2$, which is a finite sum of Dirac measures

(point masses), then the attenuation function μ can be found, up to an additive constant, from the data provided by the observed $R_\mu f$. Solmon [574] shows more in the case of the exponential Radon transform: if f is a compactly supported distribution in the plane and $T_\mu f$ is known, then μ can be determined from $T_\mu f$, if and only if f is not a radial distribution.

The identification problem is closely related to the structure of the range of R_μ on the space C_0^∞ as described in the next result.

Theorem 5.25 (Natterer [438]). *Let H be the Hilbert transform acting on the second variable of the point $(\omega, s) \in Z = S^1 \times \mathbb{R}$, let I be the identity operator and let $\varphi = \varphi(\omega)$ be the argument of the direction ω . If $\mu \in C_0^\infty(\mathbb{R}^2)$ is real valued, f is a distribution with support in the unit disc of $\mathbb{R}^2$ and $g = R_\mu f$, then for $k > m \geq 0$, k, m integers we have the consistency conditions*

$$\left\langle e^{\frac{(I+iH)}{2}(R\mu)} g, s^m e^{-ik\varphi} \right\rangle_{L^2(Z)} = 0 \quad (5.8)$$

Remark 5.26. In (5.8), $R\mu$ is the ordinary Radon transform of the attenuation function μ . The expression $e^{\frac{(I+iH)}{2}(R\mu)}$ is computed by applying the operator $\frac{1}{2}(I + iH)$ to $R\mu$, evaluating the result at (ω, s) and then taking the exponent.

Remark 5.27. The conditions (5.8) are a generalization of the consistency conditions for the classical Radon transform that we considered in chapter 4. To see this, we write out the conditions (5.8) in integral form in the case $\mu = 0$:

$$\int_0^{2\pi} e^{ik\varphi} \left(\int_{-\infty}^{\infty} s^m g(\varphi, s) ds \right) d\varphi = 0 \quad \text{for } k > m \geq 0, k, m \text{ integers.}$$

If we expand the inner integral in a Fourier series in φ , then these conditions imply that the series only has nonzero coefficients for $k \leq m$. But such a series is the restriction of a homogeneous polynomial of degree m to the unit circle. This implies that the consistency conditions are satisfied.

If the attenuation function μ is not identically zero, then the conditions (5.8) are necessary, but not sufficient in general for g to be the attenuated Radon transform $R_\mu f$ of some compactly supported function f . Recently, Novikov [454] provided necessary and sufficient consistency conditions for a function to be in the range of the attenuated Radon transform. See section 5.10.2 for related results.

Using these consistency conditions (5.8) Natterer was able to prove:

Theorem 5.28 ([438]). *Let f be a finite linear combination of at most n Dirac measures in the unit disk of $\mathbb{R}^2$. If $\mu, \nu \in C_0^\infty(\mathbb{R}^2)$ are attenuation functions such that $R_\mu f = R_\nu f$, then there is a constant C such that*

$$\mathcal{D}(\mu - \nu)(x, \omega) = C$$

for all directions ω and all $x \in \text{supp}(f)$.

Natterer also showed that if there were at least one source possessing a line missing the support of μ , then the constant C is zero and the divergent beam transforms of μ and ν are identical.

If this result could be generalized to infinitely many sources, a source being a point where we have one of the Dirac measures, then it would be possible to get a relation between μ and ν directly. But Natterer showed that this theorem is false with infinitely many sources. For example, if the source function f is radial, then the consistency conditions are satisfied for any radial attenuation function μ . In that case arbitrarily different radial attenuation functions satisfy the consistency conditions (5.8) and we cannot expect to be able to determine μ from $R_\mu f$ for radial f and radial μ .

See section 5.10.2 for a result of Boman [60] on generalized Radon transforms which is related to the identification problem discussed here.

There are many papers on the practical side of implementing SPECT tomography. A recent paper [452] by Noo and Wagner also has a large bibliography of papers on the practical side of the exponential Radon transform. Also of practical interest are the following papers: Hawkins, Yang, and Lechner [257, 258], Gullberg and Zeng [250], Liang, Ye, and Harrington [390], Metz and Pan [430], Pan, Kao, and Metz [467], Weng, Zeng, and Gullberg [621], Clarkson [98], Clack et al. [96], and Wagner, Noo, and Clackdoyle [615].

5.4 The Generalized Radon Transform on Hyperplanes

The generalized Radon transform on hyperplanes is a more universal form of the Radon transform in which the measure on each hyperplane can vary. The standard Radon transform $Rf(\theta, p)$ is determined by integration over the hyperplane $\langle x, \theta \rangle = p$ with respect to Lebesgue measure (compare, definition 2.28). If we replace Lebesgue measure by an arbitrary measure on the hyperplane, then we arrive at the following provisional definition of the generalized Radon transform on hyperplanes:

$$\mathcal{R}f(\theta, p) = \int_{\langle x, \theta \rangle = p} f(x) dv_{\theta, p}$$

where $v_{\theta, p}$ is the given measure on the hyperplane $\langle x, \theta \rangle = p$. We now make the restriction that $v_{\theta, p}$ is σ finite and absolutely continuous with respect to Lebesgue measure. By the Radon Nikodym theorem there is a nonnegative, finite valued, measurable function μ on the hyperplane $\langle x, \theta \rangle = p$ such that $dv_{\theta, p} = \mu(x, \theta, p) dx$, where dx is Lebesgue measure on this hyperplane. We assume that $v_{\theta, p} = v_{-\theta, -p}$ for the measures to depend only on the hyperplanes. The effect of this assumption is that μ is an even function in the variables θ and p . With these restrictions the provisional definition takes the following more useful form.

Definition 5.29 (Generalized Radon transform on hyperplanes). Let $\mu : \mathbb{R}^n \times S^{n-1} \times \mathbb{R} \rightarrow \mathbb{C}$ be a measurable function which is even in the last two variables. The

generalized Radon transform $\mathcal{R}_\mu$ is defined by letting

$$\mathcal{R}_\mu f(\theta, p) = \int_{\langle x, \theta \rangle = p} f(x) \mu(x, \theta, p) dx$$

for functions defined on $\mathbb{R}^n$ for which the hyperplane integrals exist.

The function μ is called the **weight** of the transform $\mathcal{R}_\mu$.

Just as we defined the formal adjoint of the Radon transform (definition 2.74), we can define the formal adjoint of the generalized Radon transform.

Definition 5.30 (Formal adjoint of the generalized radon transform).

$$\mathcal{R}_\mu^\# g(x) = \int_{S^{n-1}} g(\theta, \langle x, \theta \rangle) \mu(x, \theta, \langle x, \theta \rangle) d\theta$$

Example 5.31 (Adjoint of the attenuated and exponential Radon transforms). The attenuated Radon transform R_μ is a generalized Radon transform with weight function $\exp(-\mathcal{D}\mu(x, \theta^\perp))$, the operator $\mathcal{D}$ being the divergent beam transform: $\mathcal{D}\mu(x, \theta) = \int_0^\infty \mu(x + s\theta) ds$. Therefore,

$$R_\mu^\# g(x) = \int_{S^{n-1}} g(\theta, \langle x, \theta \rangle) \exp(-\mathcal{D}\mu(x, \theta^\perp)) d\theta.$$

The exponential Radon transform R_μ is a generalized Radon transform with weight function $\exp(\mu(x, \theta^\perp))$ and adjoint

$$T_\mu^\# g(x) = \int_{S^{n-1}} g(\theta, \langle x, \theta \rangle) e^{\mu(x, \theta^\perp)} d\theta.$$

The next result is the analogue of theorem 2.75 for the Radon-transform and theorem 3.29 for the k -plane transform. It is possible for the weight function of a generalized Radon transform to be complex valued. In the next result $\bar{\mu}$ denotes the complex conjugate of μ .

Theorem 5.32. *Let f be an integrable function defined on $\mathbb{R}^n$ and let g be an integrable function defined on $S^{n-1} \times \mathbb{R}$. Also assume that μ is a bounded measurable weight function. Then we have*

$$\langle \mathcal{R}_\mu f, g \rangle = \left\langle f, \mathcal{R}_{\bar{\mu}}^\# g \right\rangle$$

Proof. Since f and g are integrable and since μ is bounded measurable function, then all the products occurring in the integrals defining the inner products are integrable functions. Fubini's theorem therefore provides the justification for the various changes

in the order of integration that follow.

$$\begin{aligned}
 \langle f, \mathcal{R}_\mu^\# g \rangle &= \int_{\mathbb{R}^n} f(x) \overline{\mathcal{R}_\mu^\# g(x)} dx \\
 &= \int_{\mathbb{R}^n} f(x) \overline{\int_{S^{n-1}} g(\theta, \langle x, \theta \rangle) \mu(x, \theta, \langle x, \theta \rangle) d\theta} dx \\
 &= \int_{S^{n-1}} \int_{\mathbb{R}^n} f(x) \overline{g(\theta, \langle x, \theta \rangle)} \overline{\mu}(x, \theta, \langle x, \theta \rangle) dx d\theta.
 \end{aligned} \tag{5.9}$$

In the inner integral, θ is fixed so we can use proposition 2.30, the hyperplane integration theorem to obtain

$$\begin{aligned}
 \int_{\mathbb{R}^n} f(x) \overline{g(\theta, \langle x, \theta \rangle)} \overline{\mu}(x, \theta, \langle x, \theta \rangle) dx &= \int_{p \in \mathbb{R}} \int_{\langle x, \theta \rangle = p} f(x) \overline{g(\theta, p)} \overline{\mu}(x, \theta, p) dx dp \\
 &= \int_{p \in \mathbb{R}} \overline{g(\theta, p)} \int_{\langle x, \theta \rangle = p} f(x) \overline{\mu}(x, \theta, p) dx dp \\
 &= \int_{p \in \mathbb{R}} \overline{g(\theta, p)} \mathcal{R}_{\overline{\mu}} f(\theta, p) dp.
 \end{aligned}$$

Substituting this result back into equation (5.9), we get

$$\langle f, \mathcal{R}_\mu^\# g \rangle = \int_{S^{n-1}} \int_{p \in \mathbb{R}} \mathcal{R}_\mu f(\theta, p) \overline{g(\theta, p)} dp d\theta = \langle \mathcal{R}_\mu f, g \rangle.$$

■

Our aim is to study the invertibility of generalized Radon transforms. A related idea would be to derive an approximate inversion formula similar to the filtered backprojection theorem, chapter 2, theorem 2.79, which states: $R^\#(Rf * w) = f * R^\#(w)$. The analogous formula for a generalized Radon transform is

$$\mathcal{R}_{-\mu}^\#(\mathcal{R}_\mu f * w) = f * \mathcal{R}_\mu^\#(w). \tag{5.10}$$

However, this hope is dashed, in general, because Hazou and Solmon [261] have shown that a generalized Radon transform which satisfies an equation of the form (5.10) must be a Hertle exponential Radon transform, if the weight function $\mu = \mu(\theta, x)$ is continuous. More precisely, and more generally, given continuous weight functions $\mu = \mu(\theta, x)$ and $\nu = \nu(\theta, x)$, Hazou and Solmon show that the condition that for each continuous K on $S^{n-1} \times \mathbb{R}$, there exists a locally integrable E such that the equation

$$\mathcal{R}_\nu^\#(\mathcal{R}_\mu f * K) = f * E \tag{5.11}$$

is valid for all $f \in C_0^\infty \subset \mathbb{R}^n$, holds if and only if the generalized Radon transforms $\mathcal{R}_\mu$ and $\mathcal{R}_\nu$ are such that $\mathcal{R}_\mu$ is an exponential Radon transform with $\mu(\theta, x) = \exp(z(\theta) \cdot x)$, with z a continuous vector field on S^{n-1} , and $\nu(\theta, x) = \frac{\nu(\theta, 0)}{\mu(\theta, x)}$.

They also obtain Sobolev space estimates for these types of Radon transforms and an approximate inversion formula.

In the special case that $v = -\mu$, then $\mathcal{R}_\mu$ must be a Hertle exponential Radon transform and E must be $\mathcal{R}_{-\mu}^\#$. Therefore, for other species of generalized Radon transforms, we must look elsewhere than filtered back projection for inversion formulas. We refer the reader to [261] for details.

5.4.1 Local Invertibility of Generalized Radon Transforms

All the Radon transforms that we have encountered previously have been injective linear maps. This is not true for generalized Radon transforms (see section 5.4.4). However, there are many generalized Radon transforms that are invertible. We will investigate some of these presently.

The purpose of this section is to define the concept of local invertibility and to show that under only mild restrictions generalized Radon transforms are locally invertible.

Definition 5.33. Let $\mathcal{C}$ be a class of functions such that $\mathcal{C}(U)$ is defined for any nonempty open set in $\mathbb{R}^n$. For example $\mathcal{C}$ could be the class of C^∞ functions or the class of L^p functions. A generalized Radon transform $\mathcal{R}_\mu$ is said to be **locally invertible on $\mathcal{C}$** if every x in $\mathbb{R}^n$ has some neighborhood U_x such that the operator $\mathcal{R}_\mu$ is an injective operator when restricted to functions in $\mathcal{C}$ which are supported on U_x .

In their book on geometric asymptotics Guillemin and Sternberg proved the following result:

Theorem 5.34 (Guillemin and Sternberg [248]). *If the weight function μ is C^∞ , then $\mathcal{R}_\mu$ is locally invertible on rapidly decreasing functions.*

Their proof is based on showing that $\mathcal{R}_\mu^\# \mathcal{R}_\mu$ is an elliptic Fourier integral operator; the reader is referred to [248] for the details of the proof. Markoe and Quinto [424] were able to prove the following more general result by a more elementary method. We present the argument for $\mathbb{R}^2$, but as remarked in [424] the technique extends to generalized Radon transforms on $\mathbb{R}^n$.

Theorem 5.35 (Markoe and Quinto [424]).¹ *Let μ be a strictly positive C^2 weight function for the generalized Radon transform $\mathcal{R}_\mu$ on $\mathbb{R}^2$ and let $p > 2$. Then $\mathcal{R}_\mu$ is locally invertible on L^p functions. In particular, $\mathcal{R}_\mu$ is locally invertible on rapidly decreasing functions.*

Proof. We make some initial reductions and introduce some notation. First, we parametrize the direction by the angular argument, so we assume that $\mu = \mu(x, \theta, s)$ with $x \in \mathbb{R}^2$, $\theta \in \mathbb{R}$, $s \in \mathbb{R}$. Since the direction is parameterized by the angle θ , then μ is periodic of period 2π in θ . But we can show that there is no loss of generality by assuming that μ has period π in θ . For if we are given a 2π periodic weight function $\bar{\mu}$,

¹ This theorem is more general than the one stated in [424], because here we allow $\mu = \mu(x, \theta, s)$, whereas in [424] the theorem is proved for $\mu = \mu(x, \theta)$.

then we can define $\mu(x, \theta, s) = \bar{\mu}(x, \theta, s) + \bar{\mu}(x, \theta + \pi, s)$. The π periodicity of μ follows immediately from the 2π periodicity of $\bar{\mu}$ and we have

$$\mathcal{R}_\mu f(\theta, s) = \mathcal{R}_{\bar{\mu}} f(\theta, s) + \mathcal{R}_{\bar{\mu}} f(\theta + \pi, s)$$

and from this $\mathcal{R}_{\bar{\mu}} f = 0$ implies $\mathcal{R}_\mu f = 0$. The local injectivity of $\mathcal{R}_{\bar{\mu}}$ then follows from that of $\mathcal{R}_\mu$.

For any $\theta \in [0, 2\pi]$ we let $\theta^* = (\cos(\theta), \sin(\theta)) \in S^1$ and we let $\theta^\perp = (-\sin(\theta), \cos(\theta))$. Note that θ^* and $\theta^\perp$ are 2π periodic functions of θ .

There is no loss in generality by only proving local invertibility at the origin. Finally, because we are only interested in local invertibility, we can smoothly modify $\mu(x, \theta, s)$ to be 1 for x outside a compact neighborhood of the origin. In particular, this means that μ and its derivatives of order less than or equal to two will be uniformly bounded.

Define the following generalized backprojection operator: for $g \in L^p_{\text{loc}}([0, 2\pi] \times \mathbb{R})$

$$\mathcal{R}_\mu^0 g(x) = \int_0^{2\pi} \frac{g(\theta, x \cdot \theta^*)}{2\mu(x, \theta, x \cdot \theta^*)} d\theta.$$

Because μ is uniformly bounded away from zero a variation of the argument given in theorem 3.110 and its corollary shows that the operators $\mathcal{R}_\mu : L^p(B(1)) \rightarrow L^p([0, 2\pi] \times \mathbb{R})$ and $\mathcal{R}_\mu^0 : L^p([0, 2\pi] \times \mathbb{R}) \rightarrow L^p_{\text{loc}}(\mathbb{R}^2)$ are continuous for $1 \leq p \leq \infty$.

Furthermore if we define the function M by

$$M(x, y, \theta) = \frac{\mu(x - y, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp) - \mu(x, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp)}{\mu(x, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp)} \quad (5.12)$$

and if we define the operator K on compactly supported L^p functions by

$$Kf(x) = \int_{\mathbb{R}^2} \frac{f(x - y) M(x, y, \arg y)}{|y|} dy, \quad (5.13)$$

then we have the following analogue to the Fuglede theorem of chapter 3. ■

Lemma 5.36. *If $f \in L^p_0(\mathbb{R}^2)$, then*

$$\mathcal{R}_\mu^0 \mathcal{R}_\mu f = f * \left(\frac{1}{|x|} \right) + Kf(x). \quad (5.14)$$

Proof. Because we are working in $\mathbb{R}^2$ we can parametrize points on the hyperplane $\langle y, \theta^* \rangle = p$ by $y = p\theta^* + t\theta^\perp$ for $t \in \mathbb{R}$. Then

$$\mathcal{R}_\mu f(\theta, p) = \int_{-\infty}^{\infty} f(p\theta^* + t\theta^\perp) \mu(p\theta^* + t\theta^\perp, \theta, p) dt.$$

Using this and the definition of $\mathcal{R}_\mu^0$, we get

$$\begin{aligned} & \mathcal{R}_\mu^0 \mathcal{R}_\mu f(x) \\ &= \int_0^{2\pi} \left(\int_{-\infty}^{\infty} f((x \cdot \theta^*)\theta^* + t\theta^\perp) \mu((x \cdot \theta^*)\theta^* + t\theta^\perp, \theta, x \cdot \theta^*) dt \right) \\ & \quad \times \frac{1}{2\mu(x, \theta, x \cdot \theta^*)} d\theta \end{aligned}$$

Make the change of variable $t = x \cdot \theta^\perp - r$. Since $x = (x \cdot \theta^*)\theta^* + (x \cdot \theta^\perp)\theta^\perp$, then we get

$$\begin{aligned} \mathcal{R}_\mu^0 \mathcal{R}_\mu f(x) &= \int_0^{2\pi} \left(\int_{-\infty}^{\infty} f(x - r\theta^\perp) \mu(x - r\theta^\perp, \theta, x \cdot \theta^*) dr \right) \\ & \quad \times \frac{1}{2\mu(x, \theta, x \cdot \theta^*)} d\theta. \end{aligned}$$

Also make the change of variable $\theta + \frac{\pi}{2} \rightarrow \theta$, noting that $-\theta^*$ becomes $\theta^\perp$ and $\theta^\perp$ becomes θ^* under this substitution. Thus, $\mathcal{R}_\mu^0 \mathcal{R}_\mu f(x)$ becomes

$$\begin{aligned} & \int_{\frac{\pi}{2}}^{2\pi - \frac{\pi}{2}} \left(\int_{-\infty}^{\infty} f(x - r\theta^*) \mu(x - r\theta^*, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp) dt \right) \\ & \quad \times \frac{1}{2\mu(x, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp)} d\theta. \end{aligned}$$

Note that the integrand is 2π periodic because of the hypothesis on μ and the fact that the functions θ^* and $\theta^\perp$ are 2π periodic. Therefore, we can replace the outer integral by $\int_0^{2\pi}$. After doing this we can write the result as a sum of two integrals, thereby obtaining

$$\mathcal{R}_\mu^0 \mathcal{R}_\mu f(x) = I_1 + I_2 \tag{5.15}$$

where

$$\begin{aligned} I_1 &= \int_0^\pi \left(\int_{-\infty}^{\infty} f(x - r\theta^*) \mu(x - r\theta^*, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp) dt \right) \\ & \quad \times \frac{1}{2\mu(x, \theta - \frac{\pi}{2}, -x \cdot \theta^\perp)} d\theta \end{aligned}$$

and

$$I_2 = \int_{\pi}^{2\pi} \left(\int_{-\infty}^{\infty} f(x - r\theta^*) \mu\left(x - r\theta^*, \theta - \frac{\pi}{2}, -x \cdot \theta^{\perp}\right) dt \right) \\ \times \frac{1}{2\mu\left(x, \theta - \frac{\pi}{2}, -x \cdot \theta^{\perp}\right)} d\theta.$$

We can transform the second integral by making the substitution $\theta \rightarrow \theta + \pi$. This results in $\theta^* \rightarrow -\theta^*$ and $\theta^{\perp} \rightarrow -\theta^{\perp}$ and, hence,

$$I_2 = \int_0^{\pi} \left(\int_{-\infty}^{\infty} f(x + r\theta^*) \mu\left(x + r\theta^*, \theta + \frac{\pi}{2}, x \cdot \theta^{\perp}\right) dt \right) \\ \times \frac{1}{2\mu\left(x, \theta + \frac{\pi}{2}, x \cdot \theta^{\perp}\right)} d\theta.$$

After making the polar coordinate substitution $y = r\theta^*$ in the two integrals we get

$$I_1 = \int_{\mathbb{R}^2} f(x - y) \mu\left(x - y, \arg y - \frac{\pi}{2}, -(x \cdot (\arg y)^{\perp})\right) \\ \times \frac{1}{|y|} \frac{1}{2\mu\left(x, \arg y - \frac{\pi}{2}, -(x \cdot (\arg y)^{\perp})\right)} dy$$

and

$$I_2 = \int_{\mathbb{R}^2} f(x + y) \mu\left(x + y, \arg y + \frac{\pi}{2}, (x \cdot \arg y^{\perp})\right) \\ \times \frac{1}{|y|} \frac{1}{2\mu\left(x, \arg y + \frac{\pi}{2}, (x \cdot \arg y^{\perp})\right)} dy.$$

In I_2 make the substitution $y \rightarrow -y$. After taking into account that $\arg(-y) = \arg(y) - \pi$ and using the π periodicity of μ we obtain the fact that $I_1 = I_2$. Then from equation (5.15) we get

$$\mathcal{R}_{\mu}^0 \mathcal{R}_{\mu} f(x) = \int_{\mathbb{R}^2} f(x - y) \frac{\mu\left(x - y, \arg y - \frac{\pi}{2}, -(x \cdot (\arg y)^{\perp})\right)}{|y| \mu\left(x, \arg y - \frac{\pi}{2}, -(x \cdot (\arg y)^{\perp})\right)} dy.$$

On the other hand, it is easily checked that $f * (\frac{1}{|x|}) + Kf(x)$ simplifies to the right-hand side of the previous equation. ■

Lemma 5.37. *If $2 < p < \infty$ and $f \in L_0^p(\mathbb{R}^2)$, then the integrals defining $f * \frac{1}{|x|}$ and Kf converge absolutely for almost all x and $f * \frac{1}{|x|}$ and $Kf \in L^p(\mathbb{R}^2)$. Furthermore, there exists a constant d_p depending only on p , such that for all $\delta > 0$ and all $f \in$*

$L^p(B(\delta))$ we have

$$\left\| f * \frac{1}{|x|} \right\|_{L^p} \leq d_p \delta \|f\|_{L^p}. \quad (5.16)$$

Proof. Let I^α denote the Riesz potential. As we know from definition 3.55, chapter 3, $I^\alpha f$ is a constant multiple of $f * \frac{1}{|x|^{n-\alpha}}$. We now use the Hardy–Littlewood–Sobolev theorem on fractional integration (theorem 3.122) which states: if $0 < \alpha < n$, $1 < r < p < \infty$, and we have the relationship $\frac{1}{p} = \frac{1}{r} - \frac{\alpha}{n}$, then there is a constant $A_{r,p}$ dependent only on r and p , such that $\|I^\alpha f\|_{L^p} \leq A_{r,p} \|f\|_{L^r}$. We can apply this theorem to the case where $n = 2$, $\alpha = 1$, $r = \frac{2p}{2+p}$ provided that $p > 2$. This is because with the definition of r , the necessary condition $p > r > 1$ occurs if and only if $p > 2$. This is why we need the hypothesis $p > 2$ here and in theorem 5.35. Therefore, there is a constant a depending only on p such that

$$\left\| f * \frac{1}{|x|} \right\|_{L^p} \leq a \|f\|_{L^r}. \quad (5.17)$$

This argument works if $f \in L^p(\mathbb{R}^2)$ without any assumption of compact support. However, if we add the hypothesis that $f \in L^p_0(\mathbb{R}^2)$, then we can apply Hölder's inequality as follows: let $q = \frac{2+p}{p}$. Then $\frac{1}{(\frac{p}{r})} + \frac{1}{q} = 1$. Furthermore, the characteristic function χ of $\text{supp}(f)$ is in L^q , since f is compactly supported, and $f^r \in L^{\frac{p}{r}}$, since it was given that $f \in L^p$. Hölder's inequality then implies that

$$\|f\|_{L^r}^r = \int_{\mathbb{R}^2} |f(x)|^r \chi(x) dx \leq \|f^r\|_{L^{\frac{p}{r}}} \|\chi\|_{L^q}. \quad (5.18)$$

It is obvious, however, that $\|f^r\|_{L^{\frac{p}{r}}} = \|f\|_{L^p}^r$, so we get $\|f\|_{L^r} \leq \|f\|_{L^p} \|\chi\|_{L^q}^{\frac{1}{r}}$. But $\|\chi\|_{L^q}^{\frac{1}{r}} = |\text{supp}(f)|^{\frac{1}{qr}} = \sqrt{|\text{supp}(f)|}$ since $qr = 2$. This together with inequality (5.17) shows that

$$\left\| f * \frac{1}{|x|} \right\|_{L^p} \leq a \sqrt{|\text{supp}(f)|} \|f\|_{L^p} \quad (5.19)$$

which is finite since $f \in L^p$. This proves the conclusion that $f * \frac{1}{|x|} \in L^p(\mathbb{R}^2)$. It is immediate from this, the definition of the operator K and the boundedness of the weight μ that we also have $Kf \in L^p(\mathbb{R}^2)$. In particular, the integrals defining $f * \frac{1}{|x|}$ and $Kf(x)$ exist almost everywhere.

In the case that $f \in L^p(B(\delta))$, then $\sqrt{|\text{supp}(f)|} \leq \sqrt{\pi} \delta$. Using this inequality (5.19) and setting $d_p = a\sqrt{\pi}$ proves the estimate (5.16). Since a depends only on p , then this is true for d_p also. ■

If u is a distribution with L^p first derivatives, then we define the following seminorm: $\|u\|_{1,p} = \sum_{j=1}^2 \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p(\mathbb{R}^2)}$.

Lemma 5.38. *For each p with $1 < p < \infty$, there is a positive constant c_p such that for every $f \in L^p(B(1))$ we have*

$$c_p \|f\|_{L^p} \leq \left\| f * \frac{1}{|x|} \right\|_{1,p}.$$

Proof. Let R_j , $j = 1, 2$ denote the Riesz transforms on $\mathbb{R}^2$. The Riesz transform R_j is defined as the operator taking an L^2 function f into the following singular integral: $R_j f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_{|y| > \varepsilon} \frac{y_j}{|y|^3} f(x - y) dy$ (Stein [581]) and it has the properties

$$R_j f = c \cdot \frac{\partial}{\partial x_j} \left(f * \frac{1}{|x|} \right) \quad (5.20)$$

$$R_1^2 + R_2^2 = -I$$

for some constant c depending only on n ([581] and Stein and Weiss [583]). Here I denotes the identity operator. Furthermore, because the singular integral defining R_j has an odd kernel, the Calderón–Zygmund theory of singular integrals implies that $R_j : L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ is a well-defined and bounded linear map for $1 < p < \infty$ (details may be found in Stein and Weiss [583]). Hence, there is a constant b such that $\|R_j f\|_{L^p} \leq b \|f\|_{L^p}$.

Since $f \in L^p$, then the L^p boundedness of R_j implies that $R_j f \in L^p$ and equation (5.20) implies that $\frac{\partial}{\partial x_j} (f * \frac{1}{|x|}) \in L^p$. Therefore, $\|f * \frac{1}{|x|}\|_{1,p}$ is well defined and is finite.

Since $R_j f \in L^p(\mathbb{R}^2)$, then we can use the L^p boundedness of R_j to show that $\|R_j^2 f\|_{L^p} \leq b \|R_j f\|_{L^p}$. From this and (5.20) we then get $\|R_j^2 f\|_{L^p} \leq b |c| \left\| \frac{\partial}{\partial x_j} (f * \frac{1}{|x|}) \right\|_{L^p}$. Using this and the second relation (5.20) we get

$$\begin{aligned} \|f\|_{L^p} &= \left\| R_1^2 f + R_2^2 f \right\|_{L^p} \leq b |c| \cdot \sum_{j=1}^2 \left\| \frac{\partial}{\partial x_j} \left(f * \frac{1}{|x|} \right) \right\|_{L^p} \\ &= b |c| \cdot \left\| f * \frac{1}{|x|} \right\|_{1,p} \end{aligned}$$

and from this the result follows directly. ■

Lemma 5.39. *For any $2 < p < \infty$ and $c > 0$, there exists a $\delta = \delta(c, p, \mu)$ such that for all $f \in L^p(B(\delta))$ we have*

$$\|Kf\|_{1,p} \leq c \|f\|_{L^p}$$

Proof. With the conditions imposed on the weight function μ , we can get a positive lower bound on μ and uniform upper bounds on μ and its derivatives of an order less than or equal to 2. This and the mean value theorem then imply for the function M that there is a constant B_μ such that

$$\left| \frac{\partial M}{\partial x_j} \right| \leq B_\mu, \quad \left| \frac{\partial M}{\partial y_j} \right| \leq B_\mu, \quad |M| \leq B_\mu |y|, \quad \left| \frac{\partial M}{\partial \theta} \right| \leq B_\mu |y| \quad (5.21)$$

these inequalities being valid for all $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0, 2\pi]$.

Remark 5.40. Since M depends on the weight function μ , then B_μ depends on μ . Observe from these inequalities and the definition of M , that if μ is close to being a constant, then we can take B_μ close to zero.

Let any $\delta > 0$ be given. For now we assume that $f \in C_0^1(B(\delta))$. We would like to take the derivative $\frac{\partial}{\partial x_j}$ inside the integral defining Kf . To do this we look at the derivative of the integrand which is $\frac{1}{|y|} \frac{\partial M}{\partial x_j} f(x-y) + \frac{M}{|y|} \frac{\partial}{\partial x_j} f(x-y)$. The estimates (5.21) show that the absolute value of this expression is dominated by a constant multiple of $\frac{f(x-y)}{|y|}$, and lemma 5.37 shows that this is an L^1 function in the variable y , for almost every x . By the theorem on interchanging derivatives and Lebesgue integrals we can legally bring the derivative inside the integral and obtain

$$\begin{aligned} \frac{\partial}{\partial x_j} Kf(x) &= \int_{\mathbb{R}^2} \frac{\partial}{\partial x_j} \left(\frac{M(x, y, \arg y)}{|y|} \right) f(x-y) dy \\ &\quad + \int_{\mathbb{R}^2} \frac{M(x, y, \arg y)}{|y|} \frac{\partial}{\partial x_j} f(x-y) dy \end{aligned}$$

By symmetry in the $x-y$ variable we see that we can replace $\frac{\partial}{\partial x_j} f(x-y)$ by $-\frac{\partial}{\partial y_j} f(x-y)$. Once this is done we integrate by parts in the second integral and collect terms. The result is

$$\begin{aligned} \frac{\partial}{\partial x_j} Kf(x) &= \int_{\mathbb{R}^2} \left[\frac{\partial}{\partial x_j} M(x, y, \arg y) - \frac{\partial}{\partial y_j} M(x, y, \arg y) \right. \\ &\quad \left. + \frac{y_j}{|y|^2} M(x, y, \arg y) \right. \\ &\quad \left. - \frac{\partial}{\partial \theta} M(x, y, \arg y) \frac{(-1)^j y_{3-j}}{|y|^2} \right] \frac{f(x-y)}{|y|} dy. \end{aligned} \quad (5.22)$$

From the bounds in (5.21) we see that the term in square brackets is bounded by $4B_\mu$ and hence,

$$\left\| \frac{\partial}{\partial x_j} Kf \right\|_{L^p} \leq 4B_\mu \left\| |f| * \frac{1}{|x|} \right\|_{L^p}.$$

Using lemma 5.37 we see that $\|Kf\|_{1,p}$ is finite and

$$\begin{aligned} \|Kf\|_{1,p} &\leq 8B_\mu \left\| |f| * \frac{1}{|x|} \right\|_{L^p} \\ &\leq 8B_\mu d_p \delta \|f\|_{L^p}. \end{aligned}$$

If we are now given any $c > 0$ and take $\delta = \delta(p, c, \mu) = \frac{c}{8B_\mu d_p}$, then we have $\|Kf\|_{1,p} \leq c\|f\|_{L^p}$. The radius depends on μ because B_μ depends on μ and its derivatives as in remark 5.40.

The vector space $C_0^1(B(\delta))$ is dense in the Banach space $L^p(B(\delta))$. Since the inequality $\|Kf\|_{1,p} \leq c\|f\|_{L^p}$ is valid on the dense subset, then it is valid on the entire Banach space. ■

We now complete the proof of theorem 5.35. Let c_p be the constant from lemma 5.38, choose any $c < c_p$ and let $\delta = \delta(c, p, \mu)$. Then for any $f \in L^p(B(\delta))$ we have

$$\begin{aligned} \|\mathcal{R}_\mu^0 \mathcal{R}_\mu f\|_{1,p} &= \left\| f * \left(\frac{1}{|x|} \right) + Kf(x) \right\|_{1,p} \\ &\geq \left\| f * \left(\frac{1}{|x|} \right) \right\|_{1,p} - \|Kf(x)\|_{1,p} \\ &\geq (c_p - c) \|f\|_{L^p}. \end{aligned}$$

Therefore, $\mathcal{R}_\mu f = 0$ implies $\|f\|_{L^p} = 0$, and hence $f = 0$. ■

Remark 5.41. If the weight function μ is close to a constant, then the generalized Radon transform $\mathcal{R}_\mu$ is close to the usual Radon transform, which is known to be globally injective. From remark 5.40 we see that the closer the weight function μ is to a constant, the larger is the radius of the neighborhood of injectivity.

Remark 5.42. For each $x \in \mathbb{R}^n$ we have demonstrated a neighborhood of injectivity for $\mathcal{R}_\mu$ which is a ball centered at x with radius

$$\delta(c, p, \mu) = \frac{c}{8B_\mu d_p}.$$

This radius depends on $x \in \mathbb{R}^n$ because B_μ depends on the size of the derivatives of μ in a neighborhood of x . However, if μ has first derivatives that are bounded on $\mathbb{R}^n$, then we can choose a uniform B_μ and, hence, the same radius will work at every point of $\mathbb{R}^n$.

More generally, if K is a compact subset of $\mathbb{R}^n$, then we can find a single radius δ such that $\mathcal{R}_\mu$ is injective on all f with support in a disc of radius δ with center in K .

5.4.2 Rotation Invariant and Translation Invariant Radon Transforms

This section sketches the proof that rotation invariant Radon transforms are invertible. We also describe a result on the invertibility of translation invariant Radon transforms.

The concept of a rotation invariant generalized Radon transform is defined as follows.

Definition 5.43. The generalized Radon transform $\mathcal{R}_\mu$ is said to be *rotation invariant* if for every orthogonal matrix $A \in O(n)$ we have.

$$\mathcal{R}_\mu f(A\theta, p) = \mathcal{R}_\mu (f \circ A)(\theta, p)$$

Quinto [501] has proved the following result about the invertibility of rotation invariant Radon transforms.

Theorem 5.44 (Quinto [501]). Let $\mathcal{R}_\mu$ be a rotation invariant Radon transform with a C^∞ strictly positive weight function. Then

- (i) If $K > 0$ and $\mathcal{R}_\mu f$ is supported in the cylinder $\{(\theta, p) : |p| \leq K\}$, then f is supported in the closed ball $|x| \leq K$.
- (ii) $\mathcal{R}_\mu : L_0^2(\mathbb{R}^n) \rightarrow L^2(S^{n-1} \times \mathbb{R})$ is well defined and injective.

Remark 5.45. Item (i) is called the “hole theorem.” Let us punch a hole Z_1 in the cylinder $Z = S^{n-1} \times \mathbb{R}$ by removing the smaller cylinder $Z_1 = \{(\theta, p) : |p| \leq K, \theta \in S^{n-1}\}$. Suppose that two functions f, g have rotation invariant generalized Radon transforms that agree outside the hole. Then the Radon transform of their difference is supported on the cylinder Z_1 . As a consequence of item i, the functions f and g agree on the complement of the ball $|x| \leq K$. In the early days of computerized tomography, the beating heart moved too fast to be imaged by CT scanners. Hole theorems show that it is, at least theoretically, possible to use the x-ray data outside a hole containing the heart to image tissue and organs outside this hole.

Remark 5.46. Vertgeim has shown that the injectivity part of theorem 5.44 is false if the assumption of compact support is dropped. In [608] Vertgeim showed that a rotation invariant generalized Radon transform with smooth weight exists such that a nontrivial rapidly decreasing function φ exists with $\mathcal{R}_\mu \varphi = 0$. Compare example 5.66.

We provide only the main ideas of the proof. See Quinto [501] for the details.

Sketch of the proof of theorem 5.44. The fact that $\mathcal{R}_\mu : L^2_0(\mathbb{R}^n) \rightarrow L^2(S^{n-1} \times \mathbb{R})$ is well defined can be proved in exactly the same way that we proved the same result for the k -plane transform in chapter 3 (compare, corollary 3.111).

The injectivity of $\mathcal{R}_\mu$ is a direct consequence of the hole theorem, (i): if $\mathcal{R}_\mu f = 0$, then conclusion (i) implies that f is zero outside any ball centered at the origin. Hence $f = 0$ almost everywhere and we have established conclusion (ii).

It only remains to prove the hole theorem (i). The proof starts with the observation that:

- If there exists a function $U(r, p)$ defined on $\mathbb{R}^2$ which is even in both variables and for which we have the relation

$$\mu(x, \theta, p) = U(|x - p\theta|, p) \quad (5.23)$$

then $\mathcal{R}_\mu$ is rotation invariant.

In fact, if A is any orthogonal matrix, then we have

$$\mathcal{R}_\mu f(A\theta, p) = \int_{\langle x, A\theta \rangle = p} f(x) U(|x - pA\theta|, p) dx$$

Make the change of variable $x = Ay$ to obtain

$$\int_{\langle Ay, A\theta \rangle = p} f(Ay) U(|Ay - pA\theta|, p) dy = \mathcal{R}_\mu (f \circ A)(\theta, p)$$

because both the inner product and the norm are invariant under rotations. Quinto went on to prove that the converse of (5.23) is also true:

- $\mathcal{R}_\mu$ is rotation invariant if and only if there exists a function $U(r, p)$ defined on $\mathbb{R}^2$ which is even in both variables and for which we have the relation

$$\mu(x, \theta, p) = U(|x - p\theta|, p)$$

If μ is C^∞ , then so is U .

Once the nature of the weight function has been clarified, fix $f \in L_0^2(\mathbb{R}^n)$. Then there is some $M > 0$ such that $\text{supp}(f) \subset \{|x| \leq M\}$. Quinto takes the spherical harmonic expansion of f :

$$f(x) = \sum f_{lm}(|x|) Y_{lm}\left(\frac{x}{|x|}\right) \quad (5.24)$$

where for a fixed l , the functions Y_{lm} form an orthonormal basis of the space of spherical harmonics of degree l on the sphere S^{n-1} , and where each coefficient function f_{lm} satisfies $f_{lm}(r) \in L_0^2([0, M], r^{n-1} dr)$.

Quinto then considers the generalized Radon transforms $\mathcal{R}_\mu(f_{lm}(r)Y_{lm}(\theta))$, where we have set $r = |x|$, $\theta = \frac{x}{|x|}$. For $n \geq 3$, after an interesting change of coordinates and an application of the Funk–Hecke theorem, Quinto is able to show that there exist functions, denoted by $\lambda_{lm}(r)$, which act as “eigenvalues” of the generalized Radon transform $\mathcal{R}_\mu$ in the sense that

$$\mathcal{R}_\mu(f_{lm}(r)Y_{lm}(\theta)) = \lambda_{lm}(r)Y_{lm}(\theta).$$

In fact Quinto can compute these “eigenvalues”:

$$\begin{aligned} \lambda_{lm}\left(\frac{1}{t}\right) &= \frac{|S^{n-1}|}{C_l^{\frac{n-2}{2}}(1)} \int_a^t C_l^{\frac{n-2}{2}}\left(\frac{s}{t}\right) U\left(\frac{\left(1 - \left(\frac{s}{t}\right)^2\right)^{\frac{1}{2}}}{s}, \frac{1}{t}\right) \\ &\quad \times s^{-n} \left(1 - \left(\frac{s}{t}\right)^2\right)^{\frac{n-3}{2}} f_{lm}\left(\frac{1}{s}\right) ds \end{aligned} \quad (5.25)$$

In this equation, $C_l^{\frac{n-2}{2}}$ are Gegenbauer polynomials, a is a real number with $0 < a < \frac{1}{M}$. Ostensibly the integration should take place on $(0, \infty)$ for $\frac{1}{s}$ to range from 0 to ∞ . But if $s < a$, then $\frac{1}{s}$ is outside the support of f_{lm} and hence omitting the interval $(0, a)$ does not affect the value of the integral.

To prove the hole theorem (i), assume that $\mathcal{R}_\mu f$ is supported in the cylinder $\{(\theta, p) : |p| \leq K\}$. This means that $\lambda_{lm}(r) = 0$ for $r > K$. We can assume $K \leq M$ by the assumption about the support of f . If we define $b = \frac{1}{K}$, then $\lambda_{lm}(\frac{1}{t}) = 0$ for $t < b$, so the somewhat messy equation (5.25) is of the form

$$0 = \lambda_{lm}\left(\frac{1}{t}\right) = \int_a^t W(s, t) \left(1 - \left(\frac{s}{t}\right)^2\right)^{\frac{n-3}{2}} f_{lm}\left(\frac{1}{s}\right) ds \quad (5.26)$$

for $t \in [a, b]$. By the hypothesis of the theorem, μ is a strictly positive weight which results in U being strictly positive also. Since $C_l^{\frac{n-2}{2}}(1) \neq 0$, then it follows that $W(s, s) \neq 0$ for $s \in [a, b]$. If we take $n = 2$, then equation (5.26) is of the form of a generalized Abel integral equation and therefore has a unique solution. Being a homogeneous equation this gives $f_{lm}(\frac{1}{t}) = 0$ on $[a, b]$, which implies $f_{lm}(r) = 0$ for $r > K$. Similar reasoning holds if $n = 3$. In this case we have a Volterra integral equation that also has a unique solution. If $n > 3$, then equation (5.26) is neither a Volterra, nor a generalized Abel equation, but Quinto shows that a sufficient number of derivatives of

equation (5.26) results in either a Volterra or a generalized Abel equation in the same unknown. Either way, there is a unique solution and we get $f_{lm}(r) = 0$ for $r > K$. From this and the spherical harmonic expansion (5.24) we get the conclusion that $f(x) = 0$ for $|x| > K$, thus proving conclusion (i). ■

Quinto is also able to show that theorem 5.44 is true for compactly supported distributions. This result uses the theory of Fourier integral operators. He also gives several interesting applications. One application shows the injectivity of the exponential Radon transform on $\mathbb{R}^2$.

Quinto [503] has a generalization of these results to rotation invariant Radon transforms on $\mathbb{C}^n$.

We mentioned earlier, that Guillemin and Sternberg [248] have shown that if the weight function μ is C^∞ , then $\mathcal{R}_\mu^\# \mathcal{R}_\mu$ is an elliptic Fourier integral operator and as a consequence $\mathcal{R}_\mu$ is locally invertible. Following this theme, Quinto has proved:

Theorem 5.47 (Quinto [498]). *If $\mathcal{R}_\mu^\# \mathcal{R}_\mu$ is a translation invariant operator, on $\mathbb{R}^n$, then $\mathcal{R}_\mu$ is invertible. More precisely, $\mathcal{R}_\mu^\# \mathcal{R}_\mu$ is invertible by a pseudodifferential operator A and hence, $A\mathcal{R}_\mu^\#$ is a left inverse of $\mathcal{R}_\mu$. Also A is a differential operator if and only if the reciprocal of the symbol of the pseudodifferential operator $\mathcal{R}_\mu^\# \mathcal{R}_\mu$ is a polynomial in the second variable.*

Quinto's result depends on the translation invariance of $\mathcal{R}_\mu^\# \mathcal{R}_\mu$. Kurusa [372] defines a concept of translation invariance for the generalized Radon transform $\mathcal{R}_\mu$ by itself. Recall from chapter 2, that we defined $f_a(x) = f(x - a)$ to be the translation of the function f by the vector a and that theorem 2.41 showed that $Rf_a(\theta, s) = Rf(\theta, s - \langle \theta, a \rangle)$. Kurusa defines $\mathcal{R}_\mu$ to be translation invariant if $\exists v \in C^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R})$ such that

$$\mathcal{R}_\mu f_a(\theta, s) = v(a, \theta, s) \mathcal{R}_\mu f(\theta, s - \langle \theta, a \rangle).$$

If we take v to be identically 1, then by this definition the ordinary Radon transform is translation invariant.

Kurusa also defines a generalization of the Hertle exponential Radon transform (definition 5.23):

Definition 5.48 (Kurusa exponential Radon transform). A generalized Radon transform $\mathcal{R}_\mu$ is said to be a **Kurusa exponential Radon transform** if there exist smooth functions $\mu_1 : S^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$, $\mu_2 : S^{n-1} \rightarrow \mathbb{R}^n$ such that the weight satisfies the equation

$$\mu(x, \theta, p) = \mu_1(\theta, p) \exp(x \cdot \mu_2(\theta)).$$

If μ_1 is identically 1, then the Kurusa exponential Radon transform becomes the Hertle exponential Radon transform. Kurusa proves:

Theorem 5.49 (Kurusa [372]). *A generalized Radon transform $\mathcal{R}_\mu$ is translation invariant if and only if it is a Kurusa exponential Radon transform.*

We refer the reader to [372] for the proof along with other interesting results including an inversion formula for this type of transform and analogous results for the formal adjoint.

5.4.3 The Boman–Strömberg Generalized Radon Transform

In a recent paper [64], J. Boman and J. O. Strömberg devised an inversion formula for an important class of generalized Radon transforms defined on $\mathbb{R}^2$. We define a **Boman–Strömberg transform** to be a generalized Radon transform in their class; definition 5.53 contains a precise description of this class. In this paper they are also able to treat certain generalized Radon transforms with complex valued weights. Their main motivation for writing this paper was to simplify and gain insight into Novikov’s inversion theorem for the attenuated Radon transform (Novikov [453]; compare, also the discussion in section 5.3). We give Boman and Strömberg’s argument for the proof of the Novikov inversion formula.

Natterer [435], Markoe [423], and Kuchment and Shneiberg, [365] showed that an intimate connection exists between the inversion of the exponential Radon transform and complex analysis. All the inversion formulas for the attenuated Radon transform depend intimately on this connection. This will also be apparent in the development of the Boman–Strömberg inversion formula.

Finch [170] has an excellent survey of the recent advances in the study and inversion of the attenuated Radon transform.

Throughout this section Ω is an open set contained in $\mathbb{R}^2$ and all the generalized Radon transforms operate on functions living on $\mathbb{R}^2$.

Hölder Continuity

Most of the inversion formulas for the attenuated Radon transform require Hölder continuous weights. Therefore we review the idea of Hölder continuity.

Definition 5.50. Let δ be a real number, $0 < \delta \leq 1$. A function $g : \Omega \times S^1 \rightarrow \mathbb{C}$ is said to be **Hölder continuous of order δ** if a positive constant C exists such that

$$|g(y_1, \theta_1) - g(y_2, \theta_2)| \leq C |(y_1, \theta_1) - (y_2, \theta_2)|^\delta$$

for all $(y_1, \theta_1), (y_2, \theta_2) \in \Omega \times S^1$.

Similar definitions apply to functions defined on other domains X .

We use the symbol $H\ddot{o}l_\delta(X)$ to denote the class of all Hölder continuous functions defined on X of order δ .

Hölder continuous functions of order 1 are the same as Lipschitz continuous functions. Orders $\delta > 1$ are uninteresting, because a simple argument shows that in this case the function is constant on connected components of its domain.

Once we have established a value $\delta > 0$ we will merely say “ f is Hölder continuous”, the reference to δ being implicit.

Here are some elementary properties of Hölder continuous functions:

1. Sums, differences, and products of Hölder continuous functions of the same order are also Hölder continuous of the same order. This also holds for quotients if the denominator is bounded away from zero.
2. If g is Hölder continuous of order δ , then so is $|g|$.
3. If g is continuously differentiable and compactly supported, then g is Hölder continuous of order δ for any δ with $0 < \delta \leq 1$.
4. If $g : \Omega \times S^1 \rightarrow \mathbb{C}$ is Hölder continuous of order δ , then for any fixed $\theta \in S^1$ the function h defined by $h(y) = g(y, \theta)$ is Hölder continuous of order δ on Ω .
5. If g is the boundary value of an analytic function on the unit disk and if g is Hölder continuous of order δ on S^1 , then the analytic extension of g to the ball $B(1)$ is also Hölder continuous of order δ .

The proofs are elementary: Item 1 is just a variation of the sum, product, and quotient rules for derivatives. The proof of item 2 follows from the inequality $||g_1| - |g_2|| \leq |g_1 - g_2|$, whereas the proof of item 3 is a consequence of the mean value theorem for derivatives. Item 4 is obvious, whereas item 5 is a consequence of the maximum modulus principle (compare, Muskhelishvili [432], § 22).

Boundary Values of Analytic Functions

We are interested in functions $f : S^1 \rightarrow \mathbb{C}$ which extend to be analytic in the open disk $B(1)$. More precisely:

Definition 5.51. The function $f : S^1 \rightarrow \mathbb{C}$ is said to be the **boundary value of an analytic function** if a continuous function $F : \overline{B(1)} \rightarrow \mathbb{C}$ exists such that

- (1) The restriction of F to S^1 equals f and
- (2) F is analytic on the open disk $B(1)$.

We adopt the convention that the symbol f for the function on the circle also denotes the analytic extension F to the open-unit disk.

It is known that one can calculate the values of $f(z)$ for $|z| < 1$ from the boundary values. In fact $f(re^{i\theta}) = f * P_r(\theta)$, where the convolution is calculated on the circle and P_r is the Poisson kernel. In particular, if $r = 0$ we have the mean value property:

Proposition 5.52. *If f is the boundary value of an analytic function, then*

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta$$

The Boman–Strömberg Theorem

Definition 5.53. We say that $\kappa : \Omega \times S^1 \rightarrow \mathbb{C}$ is a **Boman–Strömberg weight** if

1. $\kappa \in \text{Höl}_\delta(\Omega \times S^1)$ for some real number δ with $0 < \delta \leq 1$, and $\kappa(x, \theta) > 0$,
2. there is a complex valued function $\tau \in \text{Höl}_\delta(\Omega \times S^1)$ such that $\tau(x, \theta) \neq 0$, and $\tau(x, \theta)$ is constant on all oriented lines $\langle x, \theta \rangle = p$,

3. for every fixed $x \in \Omega$, the function defined by $\theta \mapsto \kappa(x, \theta)\tau(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open unit disk $B(1) \subset \mathbb{R}^2$.

We call $\mathcal{R}_\kappa$ a **Boman–Strömberg generalized Radon transform** if κ is a Boman–Strömberg weight. The function τ is called an **associated weight** for κ .

Definition 5.54. Let $\varphi(x, \theta)$ be a function such that for every fixed $x \in \Omega$, $\theta \mapsto \varphi(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk $B(1) \subset \mathbb{R}^2$. This means that the function $\varphi(x, \theta)$ extends uniquely, in the second variable, to an analytic function on the disk $|z| < 1$. We define $\varphi(x, z)$ to be the value of this extension at z . In particular, $\varphi(x, 0)$ is just the mean value of the boundary values $\varphi(x, \theta)$.

Definition 5.55. Let $g = g(\theta, p)$ be defined on the cylinder Z . Then we define the operator Θ by

$$\Theta(g)(\theta, p) = (\theta_1 + i\theta_2)g(\theta, p).$$

J. Boman and J. O. Strömberg have shown that a Boman–Strömberg generalized Radon transform is invertible and they have given the following inversion formula. In this formula H denotes the Hilbert transform introduced in definition 3.52. The differential operator $\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$ that appears in the formula is twice the complex differential operator known as $\frac{\partial}{\partial z}$. The other operators have already been defined in this chapter.

Theorem 5.56 (Boman and Strömberg [64]). *Let κ be a Boman–Strömberg weight with associated weight τ and let*

$$m(x) = \frac{1}{2\pi} \int_0^{2\pi} \kappa(x, \theta) \overline{\tau(x, \theta)} d\theta.$$

Then we have the following inversion formula for the Boman–Strömberg generalized Radon transform: if $m(x) \neq 0$, then

$$\begin{aligned} f(x) &= \frac{1}{8\pi m(x)} \left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \right) \\ &\quad \circ \left(m(x) \left(\mathcal{R}_{1/\kappa}^\# \Theta \tau^{-1} H \tau \mathcal{R}_\kappa + \mathcal{R}_{1/\kappa}^\# \Theta \overline{\tau}^{-1} H \overline{\tau} \mathcal{R}_\kappa \right) f \right)(x). \end{aligned} \quad (5.27)$$

The symbol $\circ$ indicates application of the preceding operator.

The formula is valid pointwise if f is a Hölder continuous function. It holds in the distributional sense for complex valued L^1 functions with compact support in Ω .

We prove a few preliminary results necessary for the proof of this theorem. The proof is completed following remark 5.63.

The Boman–Strömberg theorem requires the use of several singular integrals. The most elementary singular integral is of the Cauchy principal value type and the Hilbert

transform is the simplest of this type. The following lemma shows how to regularize the Hilbert transform of Hölder continuous functions.

Lemma 5.57. *If $f \in \text{Hölder}(\mathbb{R})$ and $f \in O(|x|^{-\alpha})$ for some $\alpha > 0$, then the Hilbert transform Hf of f exists for all $x \in \mathbb{R}$ and*

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{t}{(t^2 + \varepsilon^2)} f(x - t) dt. \quad (5.28)$$

The hypotheses on f apply to any compactly supported function in $\text{Hölder}(\mathbb{R})$ and to any rapidly decreasing function defined on the real line.

Proof. The following definition of the Hilbert transform is equivalent to that given in chapter 3: $Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|t| > \varepsilon} \frac{f(x-t)}{t} dt$. The integral occurring in this definition can be expanded as

$$\begin{aligned} \int_{|t| > \varepsilon} \frac{f(x-t)}{t} dt &= \int_{\varepsilon < |t| \leq 1} \frac{f(x-t) - f(x)}{t} dt + f(x) \int_{\varepsilon < |t| \leq 1} \frac{1}{t} dt \\ &\quad + \int_{|t| > 1} \frac{f(x-t)}{t} dt. \end{aligned}$$

The last integral is finite by the growth condition on f and is independent of ε , whereas the middle integral is clearly zero. Hence, we only need to show that the first integral approaches a finite limit as $\varepsilon \rightarrow 0$. Since $f \in \text{Hölder}(\mathbb{R})$, then $\int_{|t| \leq 1} \left| \frac{f(x-t) - f(x)}{t} \right| dt \leq \int_{|t| \leq 1} \frac{C|t|^\delta}{|t|} < \infty$ which shows that $\frac{f(x-t) - f(x)}{t}$ is Lebesgue integrable on $[-1, 1]$. Then the required limit is precisely $\int_{[-1, 1]} \frac{f(x-t) - f(x)}{t} dt$. This proves that Hf exists and is finite for all $x \in \mathbb{R}$.

Because $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{f(x-t)}{t} dt$ exists, we can replace $\varepsilon \rightarrow 0$ by any sequence approaching zero. In particular, we have $Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|t| > \sqrt{\varepsilon}} \frac{f(x-t)}{t} dt$. Therefore, to prove that equation (5.28) is valid we need only establish that

$$\lim_{\varepsilon \rightarrow 0} \left| \int_{|t| > \sqrt{\varepsilon}} \frac{f(x-t)}{t} dt - \int_{\mathbb{R}} \frac{t}{(t^2 + \varepsilon^2)} f(x-t) dt \right| = 0. \quad (5.29)$$

The second integral in this limit exists since the integrand is continuous and decays like $|t|^{-1-\alpha}$ as $|t| \rightarrow \infty$. The general term of the limit simplifies to

$$\left| \int_{|t| > \sqrt{\varepsilon}} \frac{\varepsilon^2}{t(t^2 + \varepsilon^2)} f(x-t) dt - \int_{|t| \leq \sqrt{\varepsilon}} \frac{t}{(t^2 + \varepsilon^2)} f(x-t) dt \right|.$$

Because of the hypothesis on f the first integral is bounded by a constant multiple of $\int_{|t| > \sqrt{\varepsilon}} \frac{\varepsilon^2}{|t|(t^2 + \varepsilon^2)} dt$ which equals $\ln(1 + \varepsilon)$ as an elementary integration shows. This

tends to 0 as $\varepsilon \rightarrow 0$. The second integral is the same as

$$\int_{|t| \leq \sqrt{\varepsilon}} \frac{t}{(t^2 + \varepsilon^2)} (f(x-t) - f(x)) dt + f(x) \int_{|t| \leq \sqrt{\varepsilon}} \frac{t}{(t^2 + \varepsilon^2)} dt.$$

The last term is 0 because the integrand is odd, whereas the Hölder condition implies that the first term is bounded by a constant multiple of

$$\int_{|t| \leq \sqrt{\varepsilon}} \frac{|t|^{1+\delta}}{(t^2 + \varepsilon^2)} dt \leq 2 \int_0^{\sqrt{\varepsilon}} t^{-1+\delta} dt = 2 \frac{\varepsilon^{\delta/2}}{\delta}$$

which also converges to 0 as $\varepsilon \rightarrow 0$. This validates the limit in equation (5.29) and therefore completes the proof. ■

The other singular integrals that we encounter can be regularized in a similar fashion. ■

Proposition 5.58. *Let ρ be a nonzero Hölder continuous weight. For $x \neq y$ define*

$$K(x, y) = \frac{1}{\pi} \int_{S^1} \frac{\theta_1 + i\theta_2}{(x - y) \cdot \theta} \frac{\rho(y, \theta)}{\rho(x, \theta)} d\theta.$$

Then for all rapidly decreasing functions f we have

$$R_{\frac{1}{\rho}}^{\#} \ominus H R_{\rho}(f) = \int_{\mathbb{R}^2} K(x, y) f(y) dy.$$

*The function K is called a **Schwartz kernel** for the operator $R_{\frac{1}{\rho}}^{\#} \ominus H R_{\rho}$.*

Proof. Using the idea of the previous lemma, let us define $H_{\varepsilon} g(\theta, x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{(t^2 + \varepsilon^2)} g(\theta, x - t) dt$. By the various definitions we then have

$$\begin{aligned} R_{\frac{1}{\rho}}^{\#} \ominus H_{\varepsilon} R_{\rho}(f)(x) &= \int_{S^1} (\theta_1 + i\theta_2) \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{t^2 + \varepsilon^2} R_{\rho}(f)(\theta, x \cdot \theta - t) dt \right) \\ &\quad \frac{1}{\rho(x, \theta)} d\theta \\ &= \frac{1}{\pi} \int_{S^1} (\theta_1 + i\theta_2) \int_{\mathbb{R}} \frac{t}{t^2 + \varepsilon^2} \int_{y \cdot \theta = x \cdot \theta - t} f(y) \rho(y, \theta) dy dt \frac{1}{\rho(x, \theta)} d\theta. \end{aligned} \quad (5.30)$$

Let us focus attention on the two inner integrals. By the hyperplane integration theorem² we have

$$\begin{aligned} & \int_{\mathbb{R}^2} \frac{(x-w) \cdot \theta}{((x-w) \cdot \theta)^2 + \varepsilon^2} f(w) \rho(w, \theta) dw \\ &= \int_{\mathbb{R}} \int_{y \cdot \theta = s} \frac{(x-y) \cdot \theta}{((x-y) \cdot \theta)^2 + \varepsilon^2} f(y) \rho(y, \theta) dy ds. \end{aligned} \quad (5.31)$$

Note that if we make the change of variable $s = x \cdot \theta - t$, then the condition $y \cdot \theta = s$ is equivalent to $(x-y) \cdot \theta = t$ and, hence, the last equation becomes

$$\int_{\mathbb{R}} \int_{y \cdot \theta = x \cdot \theta - t} \frac{t}{t^2 + \varepsilon^2} f(y) \rho(y, \theta) dy dt = \int_{\mathbb{R}^2} \frac{(x-w) \cdot \theta}{((x-w) \cdot \theta)^2 + \varepsilon^2} f(w) \rho(w, \theta) dw$$

and equation (5.30) becomes

$$\begin{aligned} & R_{\frac{1}{\rho}}^{\#} \Theta H_{\varepsilon} R_{\rho}(f)(x) \\ &= \frac{1}{\pi} \int_{S^1} (\theta_1 + i\theta_2) \int_{\mathbb{R}^2} \frac{(x-w) \cdot \theta}{((x-w) \cdot \theta)^2 + \varepsilon^2} f(w) \rho(w, \theta) dw \frac{1}{\rho(x, \theta)} d\theta. \end{aligned}$$

Because f is rapidly decreasing and ρ is Hölder continuous, it is easy to check that the integrands are L^1 so we can apply Fubini's theorem on the product measure space $S^1 \times \mathbb{R}^2$ to obtain

$$\begin{aligned} & R_{\frac{1}{\rho}}^{\#} \Theta H_{\varepsilon} R_{\rho}(f)(x) \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \left[\int_{S^1} (\theta_1 + i\theta_2) \frac{(x-w) \cdot \theta}{((x-w) \cdot \theta)^2 + \varepsilon^2} \frac{\rho(w, \theta)}{\rho(x, \theta)} d\theta \right] f(w) dw. \end{aligned}$$

If we now take the limit as $\varepsilon \rightarrow 0$ we get

$$\begin{aligned} R_{\frac{1}{\rho}}^{\#} \Theta H R_{\rho}(f)(x) &= \int_{\mathbb{R}^2} \left[\frac{1}{\pi} \int_{S^1} \frac{(\theta_1 + i\theta_2)}{(x-w) \cdot \theta} \frac{\rho(w, \theta)}{\rho(x, \theta)} d\theta \right] f(w) dw \\ &= \int_{\mathbb{R}^2} K(x, w) f(w) dw. \end{aligned}$$

■

The next lemma is the key to getting an explicit description of the Schwartz kernels defined in proposition 5.58. We identify $\mathbb{R}^2$ with $\mathbb{C}$ in the standard way: $a = (a_1, a_2) \in$

² Proposition 2.30 in chapter 2.

$\mathbb{R}^2$ is identified with $a = a_1 + ia_2 \in \mathbb{C}$. However $a \cdot z$ still denotes the usual scalar product in $\mathbb{R}^2$.

Lemma 5.59. *Let φ be a continuous function on the closed unit disk which is analytic on the open unit disk and such that its restriction to the boundary is in $H\ddot{o}l_\delta(S^1)$.*

Let z be a nonzero complex number. Then

$$\frac{1}{\pi} \int_{S^1} \frac{\theta_1 + i\theta_2}{z \cdot \theta} \varphi(\theta) d\theta = \frac{1}{z} \left(\varphi\left(\frac{iz}{|z|}\right) + \varphi\left(-\frac{iz}{|z|}\right) \right) \text{ and} \quad (5.32)$$

$$\frac{1}{\pi} \int_{S^1} \frac{\theta_1 - i\theta_2}{z \cdot \theta} \varphi(\theta) d\theta = \frac{4}{z} \varphi(0) - \frac{1}{z} \left(\varphi\left(\frac{iz}{|z|}\right) + \varphi\left(-\frac{iz}{|z|}\right) \right) \quad (5.33)$$

The integrals are singular and are understood in a principal value sense which is made precise in the proof.

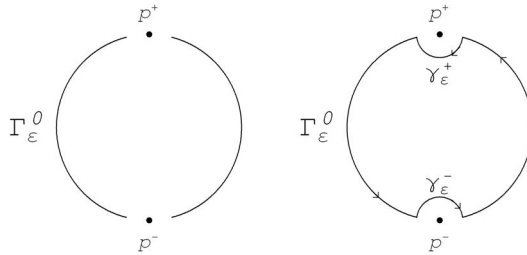
Proof. On the circle S^1 we can parametrize $\theta = \theta_1 + i\theta_2$ by $\zeta = e^{i\alpha}$, $\alpha \in [0, 2\pi]$. Then $d\theta = \frac{d\zeta}{i\zeta}$ and an easy calculation shows that $z \cdot \theta = \operatorname{Re}(z\bar{\zeta}) = \frac{1}{2}(z\bar{\zeta} + \bar{z}\zeta)$. The left-hand side of equation (5.32) can then be written as

$$I = \frac{2}{\pi} \int_{S^1} \frac{\zeta}{z\bar{\zeta} + \bar{z}\zeta} \varphi(\zeta) \frac{d\zeta}{i\zeta} = \frac{2}{\pi i} \int_{S^1} \frac{\zeta}{z + \bar{z}\zeta^2} \varphi(\zeta) d\zeta \quad (5.34)$$

since on the unit circle, $\zeta\bar{\zeta} = 1$.

The singularities in this integral occur at the two poles of the rational function $\frac{\zeta}{z + \bar{z}\zeta^2}$. It is easily checked that these poles are at $\zeta = \pm i \frac{z}{|z|}$, which lie on the unit circle. Let $p^\pm$ denote these poles.

We now excise two small arcs centered on the poles, leaving a subset of the circle denoted by Γ_ε^0 ; see the left side of the accompanying figure.



The excision is designed so that the line connecting the endpoints of the removed arc has length 2ε . Therefore we can construct a semicircle γ_ε^+ with center on the line connecting the excised endpoints and with radius equal to ε . The semicircle is chosen so that the path $\Gamma_\varepsilon^0 + \gamma_\varepsilon^+$ does not wind around the pole. Note that the distance from the pole to a point on this semicircle is greater than ε . A similar construction is performed at the other pole yielding a semicircle γ_ε^- . A look at the right side of the accompanying figure illustrates these paths and their orientations. The result is a path $\Gamma_\varepsilon^0 + \gamma_\varepsilon^+ + \gamma_\varepsilon^-$ which does not wind around either pole.

One defines the singular integral in (5.34) by

$$J = \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon^0} \frac{\zeta}{z + \bar{z}\zeta^2} \varphi(\zeta) d\zeta = \lim_{\varepsilon \rightarrow 0} J(\varepsilon). \quad (5.35)$$

where $J(\varepsilon)$ denotes the integral inside the limit. By Cauchy's theorem and the analyticity of $\frac{\zeta}{z + \bar{z}\zeta^2} \varphi(\zeta)$ inside the path $\Gamma_\varepsilon^0 + \gamma_\varepsilon^+ + \gamma_\varepsilon^-$ we see that

$$\begin{aligned} J(\varepsilon) &= - \int_{\gamma_\varepsilon^+} \frac{\zeta}{z + \bar{z}\zeta^2} \varphi(\zeta) d\zeta - \int_{\gamma_\varepsilon^-} \frac{\zeta}{z + \bar{z}\zeta^2} \varphi(\zeta) d\zeta \\ &= -J^+(\varepsilon) - J^-(\varepsilon) \end{aligned} \quad (5.36)$$

where $J^+(\varepsilon)$ and $J^-(\varepsilon)$ are the two integrals on the right, respectively.

We can evaluate the integral $J^+(\varepsilon)$ using a standard idea in the theory of singular integrals: we expand the integral as

$$\begin{aligned} J^+(\varepsilon) &= \int_{\gamma_\varepsilon^+} \frac{\zeta}{z + \bar{z}\zeta^2} (\varphi(\zeta) - \varphi(p^+)) d\zeta \\ &\quad + \varphi(p^+) \int_{\gamma_\varepsilon^+} \frac{\zeta}{z + \bar{z}\zeta^2} d\zeta \\ &= J_1^+(\varepsilon) + J_2^+(\varepsilon) \end{aligned} \quad (5.37)$$

We know that the function φ satisfies a Hölder condition of order δ on S^1 . By properties 1 and 5 of Hölder functions (following definition 5.50), this is also true on the unit disk. Hence there is a constant $c > 0$ such that on γ_ε^+

$$|\varphi(\zeta) - \varphi(p^+)| \leq c |\zeta - p^+|^\delta \leq c\varepsilon^\delta.$$

Also from the partial fraction decomposition

$$\begin{aligned} \frac{\zeta}{z + \bar{z}\zeta^2} &= \frac{1}{\bar{z}} \frac{\zeta}{\left(\zeta - i\frac{z}{|z|}\right)\left(\zeta + i\frac{z}{|z|}\right)} \\ &= \frac{1}{\bar{z}} \frac{\zeta}{(\zeta - p^+)(\zeta - p^-)} \end{aligned} \quad (5.38)$$

and the fact that $\frac{1}{\bar{z}} \frac{\zeta}{(\zeta - p^-)}$ is continuous and nonzero in a neighborhood of p^+ , we see that there is a constant M , independent of ε , such that on the arc γ_ε^+ we have

$$\left| \frac{\zeta}{z + \bar{z}\zeta^2} \right| \leq M \left| \frac{1}{\zeta - p^+} \right| \leq M\varepsilon^{-1}$$

because the distance from the pole p^+ to a point on γ_ε^+ is greater than ε . Putting these estimates together yields

$$\begin{aligned} |J_1^+(\varepsilon)| &\leq M c \varepsilon^{-1+\delta} \int_{\gamma_\varepsilon^+} d\zeta \\ &= \pi M c \varepsilon^\delta \end{aligned}$$

because $\int_{\gamma_\varepsilon^+} d\zeta$ is the length $\pi\varepsilon$ of the semicircle γ_ε^+ . In particular, $J_1^+(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

We evaluate $J_2^+(\varepsilon)$ by the theory of residues. From equation (5.38) the residue of $\frac{\zeta}{z+\bar{z}\zeta^2}$ at the pole p^+ is $\frac{1}{2\bar{z}}$. Because the winding number of γ_ε^+ around the pole p^+ is $-\frac{1}{2}$ we get

$$J_2^+(\varepsilon) = \varphi(p^+) 2\pi i \left(-\frac{1}{2}\right) \frac{1}{2\bar{z}} = -\frac{\pi i}{2\bar{z}} \varphi\left(i \frac{z}{|z|}\right).$$

Hence, $J^+(\varepsilon) = -\frac{\pi i}{2\bar{z}} \varphi\left(i \frac{z}{|z|}\right) + J_1^+(\varepsilon) \rightarrow -\frac{\pi i}{2\bar{z}} \varphi\left(i \frac{z}{|z|}\right)$ as $\varepsilon \rightarrow 0$. A similar calculation shows that $J^-(\varepsilon) \rightarrow -\frac{\pi i}{2\bar{z}} \varphi(-i \frac{z}{|z|})$. Using these results in equations (5.35) and (5.34) we obtain conclusion (5.32).

For equation (5.37) we proceed in a similar manner. The change of variable yields $\frac{1}{\pi} \int_{S^1} \frac{\theta_1 - i\theta_2}{z \cdot \theta} \varphi(\theta) d\theta = \frac{2}{\pi i} \int_{S^1} \frac{1}{\zeta(z+\bar{z}\zeta^2)} \varphi(\zeta) d\zeta$. The easily verified identity $\frac{1}{\zeta(z+\bar{z}\zeta^2)} = \frac{1}{z\zeta} - \frac{\bar{z}}{z+\bar{z}\zeta^2}$ gives $\frac{2}{\pi i z} \int_{S^1} \frac{1}{\zeta} \varphi(\zeta) d\zeta - \frac{2}{\pi i} \frac{\bar{z}}{z} \int_{S^1} \frac{\zeta}{z+\bar{z}\zeta^2} \varphi(\zeta) d\zeta$. We can evaluate the first integral by Cauchy's integral formula and we already computed the second integral in the first part of the proof. The result is $\frac{4}{z} \varphi(0) - \frac{1}{z} (\varphi(\frac{iz}{|z|}) + \varphi(-\frac{iz}{|z|}))$, which proves equation (5.33). ■

If $a \in \mathbb{R}^2$ with $a \neq 0$, then we use the symbol $a^\perp$ to denote the unit vector orthogonal to a , which is obtained by rotating a ninety degrees in the counterclockwise direction. Under the identification of $\mathbb{R}^2$ with $\mathbb{C}$ we have $a^\perp = ia$.

Proposition 5.60. *Let ρ be a nonvanishing, Hölder continuous weight of order δ such that for every fixed $x \in \Omega$, the function defined by $\theta \mapsto \rho(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk. For $x \neq y$, define*

$$Q_\rho(x, y, \eta) = \frac{\rho(y, \eta)}{\rho(x, \eta)} + \frac{\rho(y, -\eta)}{\rho(x, -\eta)}. \quad (5.39)$$

Then the operators $\mathcal{R}_{\frac{1}{\rho}}^\# \ominus H R_\rho$ and $\mathcal{R}_{1/\bar{\rho}}^\# \ominus H R_{\bar{\rho}}$ have Schwartz kernels K_+ and K_- such that

$$\mathcal{R}_{\frac{1}{\rho}}^\# \ominus H R_\rho(f)(x) = \int_{\mathbb{R}^2} K_+(x, y) f(y) dy \quad (5.40)$$

$$\mathcal{R}_{1/\bar{\rho}}^\# \ominus H R_{\bar{\rho}}(f)(x) = \int_{\mathbb{R}^2} K_-(x, y) f(y) dy \quad (5.41)$$

where

$$K_+(x, y) = \frac{1}{\bar{x} - \bar{y}} Q_\rho \left(x, y, \frac{(x - y)^\perp}{|x - y|} \right) \quad (5.42)$$

$$K_-(x, y) = \frac{1}{\bar{x} - \bar{y}} \left(4 \overline{\left(\frac{\rho(y, 0)}{\rho(x, 0)} \right)} - Q_\rho \left(x, y, \frac{(x - y)^\perp}{|x - y|} \right) \right) \quad (5.43)$$

Proof. By proposition 5.58 we have $\mathcal{R}_1^\# \ominus H \mathcal{R}_\rho(f)(x) = \int_{\mathbb{R}^2} K_+(x, y) f(y) dy$, where $K_+(x, y) = \frac{1}{\pi} \int_{S^1} \frac{\theta_1 + i\theta_2}{(x-y)^\perp} \frac{\rho(y, \theta)}{\rho(x, \theta)} d\theta$. By the hypothesis and the fundamental properties of Hölder functions, $\varphi(\theta) = \frac{\rho(y, \theta)}{\rho(x, \theta)} \in H\ddot{o}l_\delta(S^1)$. Also by hypothesis, $\varphi(\theta)$ extends to an analytic function in the unit disk. Hence, we can use lemma 5.59 with this φ and with $z = x - y$ to get

$$K_+(x, y) = \frac{1}{\bar{x} - \bar{y}} \left(\varphi \left(\frac{i(x - y)}{|x - y|} \right) + \varphi \left(-\frac{i(x - y)}{|x - y|} \right) \right).$$

By the definition of φ and the fact that for any nonzero $a \in \mathbb{R}^2$, we have $a^\perp = ia$, we immediately obtain conclusion (5.42).

Conclusion (5.43) is obtained in a similar way from the second part of lemma 5.59, by applying that lemma to the complex conjugate of $K_-(x, y)$. ■

An immediate consequence of the preceding proposition is an inversion formula for a certain class of generalized Radon transforms with complex valued weights. To prove the inversion formula we need some facts about the fundamental solution of the differential operator $\frac{\partial}{\partial \bar{z}}$.

If $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and if $P(D)$ is a differential operator, then a distribution u_0 is defined to be a **fundamental solution** of the operator $P(D)$ if $P(D)u_0 = \delta$, where δ is the Dirac distribution. It is known that if u_0 is a fundamental solution of the operator $P(D)$, then

$$P(f * u_0) = f$$

for any compactly supported distribution f on $\mathbb{R}^n$ (see Hörmander [317], sections 4.1–4.4, for details on defining convolutions of distributions and the properties of fundamental solutions).

We require the fundamental solution for the complex differential operator $\frac{\partial}{\partial \bar{z}}$, also denoted by ∂ or ∂_z . If $z = x + iy \in \mathbb{C}$, then the differential operator $\frac{\partial}{\partial \bar{z}}$ is defined by $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. We can show that a fundamental solution of $\frac{\partial}{\partial \bar{z}}$ is given by $u_0(z) = \frac{1}{\pi \bar{z}}$. We use the generalized Cauchy integral formula

$$f(\zeta) = -\frac{1}{2\pi i} \int_\gamma f(z) \frac{d\bar{z}}{\bar{z} - \bar{\zeta}} + \frac{1}{2\pi i} \iint_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{\bar{z} - \bar{\zeta}}.$$

A proof of this formula, which follows easily from the classical Stokes formula, can be found in Gunning and Rossi [251]. Here D is a domain in the complex plane

whose boundary γ is a rectifiable simple closed curve, f is a C^1 function defined on a neighborhood of D and

$$\begin{aligned} dz &= dx + i dy \\ d\bar{z} &= dx - i dy \end{aligned}$$

From these differential relations we see that

$$dz \wedge d\bar{z} = -2i dx \wedge dy. \quad (5.44)$$

We apply the generalized Cauchy integral formula in the following way: take $\zeta = 0$, let f be rapidly decreasing and take a sequence of disks D_n of radius n . Because f is rapidly decreasing, the limit of $\int_{\gamma_n} f(z) \frac{d\bar{z}}{z}$ tends to zero as $n \rightarrow \infty$, so we obtain, in the limit, $f(0) = \frac{1}{2\pi i} \iint_{\mathbb{R}^2} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z}$. The differential relation (5.44) shows that this is the same as $f(0) = \iint_{\mathbb{R}^2} \frac{1}{\pi \bar{z}} \left(-\frac{\partial f}{\partial \bar{z}}\right) dx \wedge dy$. But the value of the distribution $\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi \bar{z}}\right)$ at f is precisely $\iint_{\mathbb{R}^2} \frac{1}{\pi \bar{z}} \left(-\frac{\partial f}{\partial \bar{z}}\right) dx \wedge dy$, whereas the value of the Dirac distribution at f is $f(0)$. Hence, the last equation is equivalent to the following equality of tempered distributions: $\frac{\partial}{\partial \bar{z}}\left(\frac{1}{\pi \bar{z}}\right) = \delta$. This proves that $\frac{1}{\pi \bar{z}}$ is a fundamental solution of the $\frac{\partial}{\partial \bar{z}}$ operator. It is convenient for us to use $x = x_1 + ix_2$, instead of $z = x + iy$ as the complex variable. In this notation $\frac{1}{\pi x}$ is the fundamental solution of the operator ∂_x . As mentioned earlier, this implies

$$\partial_x \left(u * \frac{1}{\pi x} \right) = u \quad (5.45)$$

for any compactly supported distribution u .

The Boman–Strömberg inversion theorem for generalized Radon transforms with complex valued weights is:

Theorem 5.61. *Let ρ be a nonvanishing, Hölder continuous weight of order δ such that for every fixed $x \in \Omega$, the function defined by $\theta \mapsto \rho(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk, and such that $\arg \rho(x, \theta)$ is constant on all oriented lines $\langle x, \theta \rangle = p$. Let*

$$m(x) = \frac{1}{2\pi} \int_0^{2\pi} \overline{\rho(x, \theta)} d\theta.$$

Then for compactly supported C^∞ functions f we have

$$\begin{aligned} 8\pi m(x) f(x) \\ = \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(m(x) \left(\mathcal{R}_{1/\rho}^\# \ominus H \mathcal{R}_\rho + \mathcal{R}_{1/\bar{\rho}}^\# \ominus H \mathcal{R}_{\bar{\rho}} \right) f \right)(x). \end{aligned} \quad (5.46)$$

Because m may not be differentiable, this equation is meant as an equality of distributions.

Proof. The hypothesis allows us to use proposition 5.60 to compute the Schwartz kernels for $\mathcal{R}_{\frac{1}{\rho}}^\# \ominus H \mathcal{R}_\rho$ and $\mathcal{R}_{\frac{1}{\bar{\rho}}}^\# \ominus H \mathcal{R}_{\bar{\rho}}$.

It is obvious from the polar coordinate representation of complex numbers that two complex numbers with the same argument have a real quotient. Therefore, if we set $\eta = \frac{(x-y)^\perp}{|x-y|}$, then the hypothesis implies that $\frac{\rho(y, \eta)}{\rho(x, \eta)}$ is real, because $\eta \perp (x-y)$ and, hence, x and y are on the same line orthogonal to η . The same argument works for $-\eta$. Hence, the function $Q_\rho(x, y, \eta)$ in proposition 5.60 is real for the given choice of η . Adding equations (5.42) and (5.43) then gives

$$K_+(x, y) + K_-(x, y) = 4 \frac{\overline{\rho(y, 0)}}{\overline{\rho(x, 0)}} \frac{1}{\bar{x} - \bar{y}}.$$

By the hypothesis that $\rho(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk, we can use the mean value theorem for analytic functions to see that $\rho(w, 0) = \frac{1}{2\pi} \int_0^{2\pi} \rho(x, \theta) d\theta = \overline{m(w)}$. Hence, the previous equation can be written as $m(x)(K_+(x, y) + K_-(x, y)) = 4m(y)\frac{1}{\bar{x}-\bar{y}}$. If we multiply this equation by a compactly supported C^∞ function f , integrate with respect to y , and use the relations (5.40) and (5.41) of proposition 5.60, then we get

$$\begin{aligned} m(x) & \left(\mathcal{R}_{\frac{1}{\rho}}^\# \ominus H \mathcal{R}_\rho(f)(x) + \mathcal{R}_{1/\bar{\rho}}^\# \ominus H \mathcal{R}_{\bar{\rho}}(f)(x) \right) \\ & = 4 \left([m(x) f(x)] * \frac{1}{\bar{x}} \right) = 4\pi \left([m(x) f(x)] * \frac{1}{\pi \bar{x}} \right). \end{aligned}$$

Now apply the operator $(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}) = 2\partial_x$. Because $m(x)f(x)$ is a compactly supported distribution we can use the fact that $\frac{2}{\pi x}$ is a fundamental solution for this operator. Equation (5.45) then implies

$$\begin{aligned} & \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(m(x) \left(\mathcal{R}_{\frac{1}{\rho}}^\# \ominus H \mathcal{R}_\rho(f)(x) + \mathcal{R}_{1/\bar{\rho}}^\# \ominus H \mathcal{R}_{\bar{\rho}}(f)(x) \right) \right) \\ & = 8\pi \partial_x \left([m(x) f(x)] * \frac{1}{\pi \bar{x}} \right) = 8\pi m(x) f(x). \end{aligned}$$

■

We can now prove the Boman–Strömberg inversion theorem, which we restate here for convenience:

Theorem 5.56. *Let κ be a Boman–Strömberg weight with associated weight τ and let*

$$m(x) = \frac{1}{2\pi} \int_0^{2\pi} \kappa(x, \theta) \overline{\tau(x, \theta)} d\theta.$$

Then we have the following inversion formula for the Boman–Strömberg generalized Radon transform $\mathcal{R}_\kappa$: if $m(x) \neq 0$, then

$$\begin{aligned} f(x) & = \frac{1}{8\pi m(x)} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \\ & \circ \left(m(x) \left(\mathcal{R}_{1/\kappa}^\# \ominus \tau^{-1} H \tau \mathcal{R}_\kappa + \mathcal{R}_{1/\bar{\kappa}}^\# \ominus \bar{\tau}^{-1} H \bar{\tau} \mathcal{R}_{\bar{\kappa}} \right) f \right)(x) \end{aligned}$$

The formula is valid pointwise if f is a Hölder continuous function. It holds in the distributional sense for complex valued L^1 functions with compact support in Ω .

Remark 5.62. By the definition of a Boman–Strömberg weight, $\tau(x, \theta)$ is constant on all oriented lines $\langle x, \theta \rangle = p$. Therefore, we are able to define $(\tau \mathcal{R}_\kappa) f(\theta, p)$ to be the value of τ on the oriented line $\langle x, \theta \rangle = p$ times the value of the generalized Radon transform $\mathcal{R}_\kappa$, on that line.

Remark 5.63. The advantage of this inversion formula over that of theorem 5.61 is that we only require input data from the single Radon transform $\mathcal{R}_\kappa f$, whereas theorem 5.61 requires the data from two Radon transforms.

Proof. First we prove the result for compactly supported C^∞ functions f .

Define $\rho(x, \theta) = \kappa(x, \theta)\tau(x, \theta)$. By the definition of ρ , $H\mathcal{R}_\rho f(\theta, p) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{t} \int_{y \cdot \theta = p-t} f(y) \kappa(y, \theta) \tau(y, \theta) dy dt$. But we know that the associated weight τ is constant on all oriented lines. Since the set of y satisfying $y \cdot \theta = p - t$ is such a line, then we can factor $\tau(y, \theta)$ out of the inner integral, thus arriving at $H\mathcal{R}_\rho f(\theta, p) = H\tau \mathcal{R}_\kappa f(\theta, p)$. A similar result holds for the conjugates. From this an easy calculation shows that

$$\mathcal{R}_{1/\rho}^\# \Theta H \mathcal{R}_\rho = \mathcal{R}_{1/\kappa}^\# \Theta \tau^{-1} H \tau \mathcal{R}_\kappa \quad \text{and} \quad (5.47)$$

$$\mathcal{R}_{1/\bar{\rho}}^\# \Theta H \mathcal{R}_{\bar{\rho}} = \mathcal{R}_{1/\kappa}^\# \Theta \bar{\tau}^{-1} H \bar{\tau} \mathcal{R}_\kappa. \quad (5.48)$$

By the definition of a Boman–Strömberg weight, $\kappa(x, \theta)$ is real and $\tau(x, \theta)$ is constant on all oriented lines $\langle x, \theta \rangle = p$. It immediately follows that $\arg \rho(x, \theta)$ is constant on all oriented lines $\langle x, \theta \rangle = p$. Also by the definition of a Boman–Strömberg weight it follows that $\theta \mapsto \rho(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk. Everything is also Hölder continuous, so the hypotheses of theorem 5.61 apply to the weight ρ and hence the conclusion is true. Taking into account equations (5.47) and (5.48) equation (5.46) of theorem 5.61 becomes the conclusion of the current theorem, for compactly supported C^∞ functions f . We refer the reader to Boman and Strömberg [64] for the argument required when $f \in L^1$ or when f is Hölder continuous. ■

The reader has undoubtedly wondered how one comes by examples of Boman–Strömberg Radon transforms. Their motivation for introducing such transforms was to study the attenuated Radon transform, and it is their proof of Novikov’s inversion formula [453] for the attenuated Radon transform that shows how to come up with some Boman–Strömberg transforms.

Let us consider the attenuated Radon transform

$$R_\mu f(\theta, s) = \int_{\langle x, \theta \rangle = s} f(x) e^{-\mathcal{D}\mu(x, \theta^\perp)} dx$$

which is a generalized Radon transform with weight function $\exp(-\mathcal{D}\mu(x, \theta^\perp))$. The operator $\mathcal{D}$ is the divergent beam transform: $\mathcal{D}\mu(x, \theta) = \int_0^\infty \mu(x + s\theta) ds$.

Decompose $\mathcal{D}\mu(x, \theta^\perp)$ into even and odd parts u and w , with respect to the second variable:

$$\begin{aligned} u(x, \theta) &= \frac{1}{2} \left(\mathcal{D}\mu(x, \theta^\perp) + \mathcal{D}\mu(x, -\theta^\perp) \right), \\ w(x, \theta) &= \frac{1}{2} \left(\mathcal{D}\mu(x, \theta^\perp) - \mathcal{D}\mu(x, -\theta^\perp) \right). \end{aligned}$$

Conjugate functions allow one to extend a continuous function on the circle to be analytic on the disk. Here is a brief introduction to conjugate functions. If v is a continuous function defined on the circle S^1 , then there is a function $\tilde{v}$ defined on S^1 such that $v(\theta) + i\tilde{v}(\theta)$ is the boundary value of a holomorphic function on the unit disk and the mean value of $\tilde{v}$ over S^1 is zero. The function $\tilde{v}$ is called the **conjugate function** of v . Zygmund [630] or Hoffman [314] or [315] are good references for this idea. Letting $\theta(\alpha) = (\cos \alpha, \sin \alpha)$, it can be shown that

$$\tilde{v}(\theta(\alpha)) = \frac{1}{2\pi} \int_0^{2\pi} v(\theta(\alpha - \beta)) \cot \frac{\beta}{2} d\beta \quad (5.49)$$

(Hoffman [314], page 79, has a particularly nice derivation of this equation; compare, Zygmund [630], page 131, or Fatou [164] pages 358–364). Also important for us is Privalov's theorem that states: $v \in H\ddot{o}l_\delta \implies \tilde{v} \in H\ddot{o}l_\delta$ (Privalov [496], compare Zygmund [630], page 121 or Lu [404], page 28).

We now show that the following six properties are true: if $\mu \in H\ddot{o}l_\delta$, then

1. $\mathcal{D}\mu(x, \theta^\perp) = u(x, \theta) + w(x, \theta)$ and w and $\tilde{w}$ are odd in the θ variable.
2. For each fixed x , $w(x, \theta) + i\tilde{w}(x, \theta)$ is the boundary value of a nonvanishing analytic function defined in the open-unit disk.
3. u, w , and $\tilde{w} \in H\ddot{o}l_\delta$.
4. $u(x, \theta) = \frac{1}{2} R\mu(\theta, x \cdot \theta)$. Here, R is the ordinary Radon transform of μ .
5. $\tilde{w}(x, \theta) = -\frac{1}{2} H R\mu(\theta, x \cdot \theta)$.
6. $u(x, \theta)$ and $\tilde{w}(x, \theta)$ are constant on all lines of the form $x \cdot \theta = p$.

Properties 1 and 2 have been established by the preceding discussion (the fact that $\tilde{w}$ is odd in θ is a consequence of formula (5.49).

Observe that $\mu \in H\ddot{o}l_\delta$ easily implies the same for the divergent beam transform $\mathcal{D}\mu(x, \theta^\perp)$. From this we get u and $w \in H\ddot{o}l_\delta$. Privalov's theorem with parameters now implies that $\tilde{w} \in H\ddot{o}l_\delta$ and this establishes property 3.

Property 4 is true because $\mathcal{D}\mu(x, \theta^\perp)$ and $\mathcal{D}\mu(x, -\theta^\perp)$ are formed by integrating μ over opposite rays emanating from x and, hence, their sum is the integral over the corresponding line formed from these rays. It is, in fact, the Radon transform $R\mu(\theta, p)$ where p is such that x lies on the line defined by the equation $x \cdot \theta = p$. In other words, $\mathcal{D}\mu(x, \theta^\perp) + \mathcal{D}\mu(x, -\theta^\perp) = R\mu(\theta, x \cdot \theta)$. Property 4 is now a direct consequence of the definition of u . Also property 4 immediately implies half of property 6 because $u(x, \theta) = \frac{1}{2} R\mu(\theta, x \cdot \theta)$ and $x \cdot \theta$ is constant on any line of the form $x \cdot \theta = p$. For

property 5 observe that

$$\begin{aligned} HR\mu(\theta, x \cdot \theta) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x \cdot \theta - p} \int_{y \cdot \theta = p} \mu(y) dy dp \\ &= \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\mu(y)}{(x - y) \cdot \theta} dy \end{aligned}$$

The last step followed by an argument similar to that in proposition 5.58, in particular, equation (5.31). Use the polar coordinate transformation $y = x + r\omega$ on the last integral to get

$$\begin{aligned} HR\mu(\theta, x \cdot \theta) &= \frac{1}{\pi} \int_{S^1} \int_0^\infty \frac{\mu(x + r\omega)}{(-r\omega) \cdot \theta} r dr d\omega \\ &= -\frac{1}{\pi} \int_{S^1} \frac{1}{\omega \cdot \theta} \mathcal{D}\mu(x, \omega) d\omega. \end{aligned}$$

Let $\omega = (\cos \gamma, \sin \gamma)$ and let $\theta = (\cos \alpha, \sin \alpha)$. Then $\frac{1}{\omega \cdot \theta} = \sec(\alpha - \gamma)$. We can then use the identity

$$\sec \psi = \frac{1}{2} \left(\cot \left(\frac{\psi + \frac{\pi}{2}}{2} \right) - \cot \left(\frac{\psi - \frac{\pi}{2}}{2} \right) \right)$$

to get

$$\begin{aligned} HR\mu(\theta, x \cdot \theta) &= -\frac{1}{2\pi} \int_0^{2\pi} \cot \left(\frac{\alpha - \gamma + \frac{\pi}{2}}{2} \right) \mathcal{D}\mu(x, \theta(\gamma)) d\gamma \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \cot \left(\frac{\alpha - \gamma - \frac{\pi}{2}}{2} \right) \mathcal{D}\mu(x, \theta(\gamma)) d\gamma. \end{aligned}$$

In the first of these integrals substitute $\beta = \alpha - \gamma + \frac{\pi}{2}$ and in the second substitute $\beta = \alpha - \gamma - \frac{\pi}{2}$. By using formula (5.49) for the conjugate function, this gives

$$\begin{aligned} HR\mu(\theta, x \cdot \theta) &= -\frac{1}{2\pi} \int_0^{2\pi} \cot \left(\frac{\beta}{2} \right) \mathcal{D}\mu \left(x, \theta \left(\alpha - \beta + \frac{\pi}{2} \right) \right) d\beta \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \cot \left(\frac{\beta}{2} \right) \mathcal{D}\mu \left(x, \theta \left(\alpha - \beta - \frac{\pi}{2} \right) \right) d\beta \\ &= -\widetilde{\mathcal{D}}\mu \left(x, \theta \left(\alpha + \frac{\pi}{2} \right) \right) + \widetilde{\mathcal{D}}\mu \left(x, \theta \left(\alpha - \frac{\pi}{2} \right) \right). \end{aligned}$$

From the fact that $\theta(\alpha)^\perp = \theta(\alpha + \frac{\pi}{2})$, $-\theta(\alpha)^\perp = \theta(\alpha - \frac{\pi}{2})$ and that any $\theta^\perp$ on the unit circle is of the form $\theta(\alpha)^\perp$ for some angle α , we get

$$\begin{aligned} HR\mu(\theta, x \cdot \theta) &= -\widetilde{\mathcal{D}\mu}(x, \theta^\perp) + \widetilde{\mathcal{D}\mu}(x, -\theta^\perp) \\ &= -2\widetilde{w}(x, \theta). \end{aligned}$$

This proves property 5 and an immediate consequence is that $\widetilde{w}(x, \theta)$ is constant on all lines of the form $x \cdot \theta = p$, thus establishing the last part of property 6.

We can now state and prove Novikov's inversion formula for the attenuated Radon transform.

Theorem 5.64 (Novikov's inversion theorem). *Let $\mu(x, \theta)$ be a positive, compactly supported, Hölder continuous attenuation function for the attenuated Radon transform R_μ . Then the following inversion formula for R_μ is valid in the sense of distributions for complex valued L^1 functions with compact support in Ω and is valid pointwise if f is a Hölder continuous function. Let*

$$\tau = \exp\left(\frac{1}{2}[R\mu(\theta, x \cdot \theta) + i(HR\mu)(\theta, x \cdot \theta)]\right),$$

then

$$f(x) = \frac{1}{8\pi} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left[\left(R_{-\mu}^\# \Theta \tau^{-1} H \tau R_\mu + R_{-\mu}^\# \Theta \bar{\tau}^{-1} H \bar{\tau} R_\mu \right) f \right](x). \quad (5.50)$$

Remark 5.65. The attenuated Radon transform $R_\mu f$ is the generalized Radon transform $\mathcal{R}_\kappa$ with weight function $\kappa(x, \theta) = \exp(-\mathcal{D}\mu(x, \theta^\perp))$. See example 5.31 for a computation of $R_{-\mu}^\#$. It is easy to check that $R_{-\mu}^\# = \mathcal{R}_{\frac{1}{\kappa}}^\#$.

Proof. Using the previous notation and results, define

$$\kappa(x, \theta) = \exp\left(-\mathcal{D}\mu(x, \theta^\perp)\right).$$

Then the hypothesis implies that $\kappa \in H\ddot{o}l_\delta(\Omega \times S^1)$ and it is obvious that $\kappa(x, \theta) > 0$, so κ satisfies the first condition of a Boman–Strömberg weight. The function τ defined in the theorem is clearly nonzero everywhere. By properties 4, 5, and 6, we see that

$$\frac{1}{2}[R\mu(\theta, x \cdot \theta) + i(HR\mu)(\theta, x \cdot \theta)] = u(x, \theta) - i\widetilde{w}(x, \theta)$$

and this function is constant on all lines of the form $x \cdot \theta = p$ and is in $H\ddot{o}l_\delta$. Hence, τ which is the exponent of this expression has the same property. Therefore, τ satisfies the second condition of a Boman–Strömberg weight.

By properties 1, 4, and 5 the expression

$$\kappa(x, \theta) \tau(x, \theta) = \exp\left(-\mathcal{D}\mu(x, \theta^\perp) + \frac{1}{2}\left[R\mu(\theta, x \cdot \theta) + \frac{1}{2}i(HR\mu)(\theta, x \cdot \theta)\right]\right).$$

transforms into

$$\kappa(x, \theta) \tau(x, \theta) = \exp(-(w(x, \theta) + i\tilde{w}(x, \theta))). \quad (5.51)$$

and hence by property 2 we see that $\kappa(x, \theta) \tau(x, \theta)$ is the boundary value of a non-vanishing analytic function defined in the open-unit disk. This verifies the remaining condition for a Boman–Strömberg weight. Therefore, we can apply theorem 5.56.

In applying theorem 5.56 recall from the remark above that $R_\mu = \mathcal{R}_\kappa$ and $R_{-\mu} = \mathcal{R}_{\frac{1}{\kappa}}$. We also need to examine $m(x)$: since κ is real, then by the definition of m and by equation (5.51) we have

$$\begin{aligned} \overline{m(x)} &= \frac{1}{2\pi} \int_0^{2\pi} \kappa(x, \theta) \tau(x, \theta) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \exp(-(w(x, \theta) + i\tilde{w}(x, \theta))) d\theta. \end{aligned}$$

For a fixed x , this is the mean value around the circle of the function

$$\exp(-(w(x, z) + i\tilde{w}(x, z)))$$

which is continuous on the circle $|z| = 1$ and analytic on the disk $|z| < 1$. Therefore, by the mean value theorem for analytic boundary value functions,

$$\overline{m(x)} = \exp(-(w(x, 0) + i\tilde{w}(x, 0))). \quad (5.52)$$

The function $w(x, z) + i\tilde{w}(x, z)$ is also continuous on the circle $|z| = 1$ and analytic on the disk $|z| < 1$ and hence

$$w(x, 0) + i\tilde{w}(x, 0) = \frac{1}{2\pi} \int_0^{2\pi} w(x, \theta) + i\tilde{w}(x, \theta) d\theta.$$

But both w and $\tilde{w}$ are odd so the previous integral is zero. Substituting this result in equation (5.3) gives $m(x) = 1$ for all $x \in \Omega$. With these remarks theorem 5.56 becomes

$$f(x) = \frac{1}{8\pi} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right) \left(\left(R_{-\mu}^\# \Theta \tau^{-1} H \tau R_\mu + R_{-\mu}^\# \Theta \bar{\tau}^{-1} H \bar{\tau} R_\mu \right) f \right)(x)$$

which is exactly the conclusion of this theorem. ■

Boman and Strömberg have several other interesting results that the reader may want to refer to. They prove a limited-angle result for Boman–Strömberg transforms (in the special case of the attenuated Radon transform, this was originally due to Novikov, op. cit.). They also discuss more concrete characterizations of Boman–Strömberg transforms and the types of such transforms which lead to the attenuated Radon transform.

Kunyansky [369] uses the Novikov inversion formula for the attenuated Radon transform to implement a SPECT reconstruction algorithm on a computer. He does some actual numerical reconstructions by using phantoms and he has a brief but useful

description of the Novikov formula. Guillement et al. [242] implement the Novikov formula and apply their algorithm to real SPECT data. Also of interest are Bal [33] and Bal and Moireau [34].

5.4.4 Noninvertible Generalized Radon Transforms

There are generalized Radon transforms that are not invertible. This is interesting in view of the wide range of locally invertible generalized Radon transforms and the few classes of invertible generalized Radon transforms that we have studied.

It is trivial to obtain noninvertible generalized Radon transforms if we allow the weight function to be zero on a large set (for a really trivial example, let it be zero for all the arguments). Therefore, we will present an example for which the weight function is positive.

Example 5.66 (Markoe and Quinto [424]). A noninvertible, rotation invariant generalized Radon transform on $\mathbb{R}^2$ with a strictly positive, bounded weight function.

Construction. We first produce an example that is not smooth but satisfies the other requirements. Define annuli $T_j = \{x : \frac{j-1}{j} \leq |x| < \frac{j}{j+1}\}$. Let $0 < |s| < 1$. Let L_s represent any straight line that is at a distance s from the origin. Let $j_0(s, L_s)$ be the minimum index j for which the line L_s intersects T_j and let $a_j(s, L_s)$ be the length of the segment $L_s \cap T_j$. Hence, for any line L_s at distance s from the origin we have $L_s \cap T_{j_0} \neq \emptyset$, but $L_s \cap T_j = \emptyset$ for $j < j_0(s)$. By the rotational symmetry of the annuli T_j , both functions j_0 and a_j are functions only of s .

Fixing an x with $|x| < 1$ and a $\theta \in [0, 2\pi]$, we let $j_0 = j_0(x \cdot \theta^*)$, where $\theta^* = (\cos \theta, \sin \theta)$. Noting that x must be in some annulus T_j , we define

$$\mu(x, \theta, s) = \begin{cases} \frac{1}{2} & \text{if } j - j_0 \text{ is even} \\ \frac{a_{j-1}(x \cdot \theta^*)}{a_j(x \cdot \theta^*)} & \text{if } j - j_0 \text{ is odd} \end{cases}$$

We also extend μ to be identically 1 when $|x| \geq 1$. Because μ has no dependence on the last variable s , we will omit that variable in the future.

It is clear that μ is positive and rotation invariant. The weight function μ is also bounded. To see this note that μ is bounded by the maximum of the number 1 and the maximum of functions of the form $\frac{a_{j-1}(s)}{a_j(s)}$ on the interval $[0, \frac{j-1}{j}]$. The values of $a_j(s)$ are easy to compute and a somewhat messy elementary calculus computation shows that $\frac{a_{j-1}(s)}{a_j(s)}$ attains a maximum on the interval $[0, \frac{j-1}{j}]$ at $s_1 = \frac{j-2}{j-1}$. Then we compute $\lim_{j \rightarrow \infty} \frac{a_{j-1}(s_1)}{a_j(s_1)} = \sqrt{2} + 1$ and this shows that μ is bounded.

We can now construct a nontrivial function in the kernel of $\mathcal{R}_\mu$. Define $f(x) = (-2)^j$ on T_j and $f(x) = 0$ for $|x| \geq 1$. It is obvious from the construction that $\mathcal{R}_\mu f(\theta, s) = 0$ if $|s| \geq 1$ since any line of distance s from the origin will not meet the union of the annuli T_j . We now consider the case $|s| < 1$. In the following calculation we let $j_0 = j_0(s)$ and we let $l(U)$ denote the one-dimensional Lebesgue measure of U . Now

$\mathcal{R}_\mu f(\theta, s) = \int_{\langle x, \theta \rangle = s} f(x) \mu(x, \theta) dx$. We only need to compute this on $|x| < 1$. On this set we can pair adjacent annuli and thereby obtain

$$\begin{aligned} \mathcal{R}_\mu f(\theta, s) &= \sum_{k=0}^{\infty} \int_{L_s \cap T_{j_0+2k}} f(x) \mu(x, \theta) dx + \int_{L_s \cap T_{j_0+2k+1}} f(x) \mu(x, \theta) dx \\ &= \sum_{k=0}^{\infty} \left[(-2)^{j_0+2k} \left(\frac{1}{2} \right) l(L_s \cap T_{j_0+2k}) \right. \\ &\quad \left. + (-2)^{j_0+2k+1} \frac{a_{j_0+2k}(s)}{a_{j_0+2k+1}(s)} l(L_s \cap T_{j_0+2k+1}) \right] \end{aligned}$$

But $l(L_s \cap T_j) = a_j(s)$ so the k th term of the sum is

$$(-2)^{j_0+2k} \left(\frac{1}{2} \right) a_{j_0+2k}(s) + (-2)^{j_0+2k+1} \frac{a_{j_0+2k}(s)}{a_{j_0+2k+1}(s)} a_{j_0+2k+1}(s) = 0.$$

Remark 5.67. As remarked in Markoe and Quinto [424], although the function f constructed in the example is not smooth, one can carefully modify both f and μ to make f a compactly supported C^∞ function, whereas μ is still a strictly positive, bounded weight function. Because of theorem 5.44, μ cannot be C_∞ . Compare remark 5.46.

Remark 5.68. Boman [62] has an example of non invertibility, on $\mathbb{R}^2$, in which both the function f and the weight μ are C^∞ . The weight is also strictly positive. This is in counterpoint to the result of Boman [59] that generalized Radon transforms with real-analytic weight functions are invertible. A more accessible proof of this result is in Boman and Quinto [63].

5.5 Generalized Radon Transforms on Other Spaces

Because of limitations of space, only a brief survey of this interesting and useful topic can be given here. It would probably take another complete volume to do full justice to this theme. The curious reader can refer to the cited articles and texts for complete details.

In section 5.4 we saw how to define a type of Radon transform on hyperplanes in which the usual Lebesgue measure on each hyperplane is replaced by a more arbitrary one. The hyperplanes in $\mathbb{R}^n$ form a family of submanifolds of $\mathbb{R}^n$, so a natural generalization would be to develop a Radon transform that integrates over certain families of submanifolds more arbitrary than the family of hyperplanes. Indeed the English translation of the title of Radon's 1917 paper [508] is "On the Determination of Functions by the Values of their Integrals over Certain Manifolds." This idea was explored by Funk even before Radon's 1917 paper. In 1916 Funk [184] showed how to determine an even function on S^2 by its integrals over great circles. In this case the ambient manifold is S^2 , not $\mathbb{R}^n$ and the family of submanifolds is the family of great circles. Later, in the

1930s, John determined functions from their line and sphere integrals in $\mathbb{R}^3$ [322, 323, and 324].

In 1966 Helgason [270] developed a very general method of generating Radon transforms on homogeneous spaces. The idea is to generalize the duality between points and hyperplanes which we find in the standard Radon transform. The type of duality we are interested in is analogous to the formal adjoint theorem (theorem 3.29, chapter 3) which in the case of the Radon transform states that

$$\left\langle f, R^\# g \right\rangle_{\mathbb{R}^n} = \langle Rf, g \rangle_{G_{n-1,n}} \quad (5.53)$$

This form of duality is reminiscent of the duality between an operator and its adjoint. The duality goes even further since the value of the Radon transform Rf at a hyperplane ξ is formed by integrating f over points x contained in the hyperplane ξ , whereas the value of the formal adjoint $R^\# g$ at a point x is found by integrating g over hyperplanes containing x . This leads to the concept of incidence introduced by Chern [88] in 1942: the point x is *incident* to the hyperplane ξ if and only if ξ contains x . Helgason generalized this notion of incidence for elements of two related homogeneous spaces. He defined a Radon transform and formal adjoint on these spaces and he showed that the formal adjoint relationship (5.53) is valid in this more general context.

The unifying idea is to find a single locally compact topological group G and to find two closed subgroups H and K such that the homogeneous spaces $X = G/H$ and $\Xi = G/K$ have a dual relationship in the sense just described. The trick, and the beauty, of this idea is that a single group G can have homogeneous spaces X and Ξ as different as $X = \mathbb{R}^n$ and $\Xi =$ the manifold of hyperplanes in $\mathbb{R}^n$ such that the desired duality holds. We will show that one can find a group G that models the classical Radon transform. We then briefly describe a few more general situations of the same nature. Before doing this, let us investigate the abstract situation.

Let G be a locally compact topological group and let H and K be two subgroups. Define homogeneous spaces $X = G/H$ and $\Xi = G/K$. Then the elements of X (resp. Ξ) are the left cosets of G modulo H (resp. modulo K). Helgason defines the notion of incidence as follows: cosets $x \in G/H$ and $\xi \in G/K$ are incident if $x \cap \xi \neq \emptyset$ (recall that cosets are subsets of G , so the intersection makes sense).

Because we want to define Radon transforms we need measures, in particular, Haar measures. Therefore, we formalize the situation with the following definition:

Definition 5.69. Spaces X and Ξ are said to be *homogeneous spaces in duality* if a locally compact topological group G with closed subgroups H and K exists such that $X = G/H$, $\Xi = G/K$ and the groups G , H , K , and $H \cap K$ are unimodular.

It is the assumption of unimodularity that allows us to produce the necessary Haar measures. This assumption is satisfied, for all the groups in question, if G is compact or Abelian.

For each $x \in X$, Helgason lets

$$\begin{aligned}\check{x} &= \{\xi \in \Xi : x \text{ is incident to } \xi\} \\ \hat{x} &= \{x \in X : \xi \text{ is incident to } x\}.\end{aligned}$$

Helgason shows that each $\check{x}$ is closed and is a translate of a fixed orbit $\check{x}_0$, the same being true of $\hat{x} \in \Xi$ with respect to a fixed orbit $\hat{x}_0$. In fact, one can take $\check{x}_0$ to be the collection of cosets of the form hK , $h \in H$. Letting $L = H \cap K$, it is then easy to show that $\check{x}_0$ is homeomorphic to H/L . The same reasoning shows that $\hat{x}_0$ is homeomorphic to K/L . Also, the following fact is easy to demonstrate.

Lemma 5.70. *$x = \gamma H$ is incident to $\xi = gK$ if and only if $g^{-1}\gamma \in KH$, where KH is the set of all products of elements of K by elements of H .*

Corollary 5.71. *If $\xi = gK \in \Xi$, then $\hat{\xi} = \{gkH : k \in K\}$. Furthermore, we have a one-to-one correspondence $\hat{\xi} \simeq K/L$. By duality, if $x = \gamma H \in X$, then $\check{x} = \{\gamma hK : h \in H\}$ and $\check{x} \simeq H/L$.*

Proof. Assume that $x \in \hat{\xi}$. By the lemma, $x = \gamma H$, where $g^{-1}\gamma \in KH$. Hence, $\exists k \in K$ and $h \in H$ such that $g^{-1}\gamma = kh$ or $\gamma = gkh$. Since $h \in H$, then the coset $gkhH$ is the same as the coset gkH and hence $x = gkH$. Conversely, if $x = gkH$ for some $k \in K$, then if we let $\gamma = gk$, then $g^{-1}\gamma = k \in KH$ and by the lemma $x \in \hat{\xi}$.

Let us define a function $\Phi : \hat{\xi} \rightarrow K/L$ as follows: for any $gkH \in \hat{\xi}$, let $\Phi(gkH) = kL$. This is well defined because gkH defines the same coset as gk_1H if and only if $k^{-1}k_1 = (gk)^{-1}gk_1 \in H$. But each k factor is in the group K , so $(gk)^{-1}gk_1 \in L = H \cap K$. Therefore, $\Phi(gkH) = \Phi(gk_1H)$. A similar calculation shows that the correspondence is injective and surjective. ■

We now suppose that the homogeneous spaces H/L and K/L have left invariant Haar measures dh and dk , respectively.

We want to define the Radon transform Rf of a compactly supported continuous function f defined on X . The value of Rf at a dual point $\xi \in \Xi$ will be defined as the integral of f over all points x incident to ξ ; namely, it will be the integral over $\hat{\xi}$. The dual Radon transform is defined in a similar way. By the corollary, to form the Radon transform we should integrate $f(gkH)$ over K/L . This leads to the following definition of the Helgason Radon transform:

Definition 5.72. The value of the *Helgason Radon transform* of a compactly supported continuous function f on the homogeneous space $X = G/H$ is defined as a function on the dual homogeneous space $\Xi = G/K$ as follows: for $\xi = gK \in \Xi$ define

$$Rf(\xi) = Rf(gK) = \int_{k \in K/L} f(gkH) dk.$$

The dual transform R^t is defined on X as follows: for $x = \gamma H \in X$ define

$$R^t \phi(x) = \int_{h \in H/L} f(\gamma h K) dh.$$

Since $\hat{\xi}$ is identified with K/L and $\check{x}$ is identified with H/L , then we can write the previous definition as

$$Rf(\xi) = \int_{\check{x}} f(x) dx$$

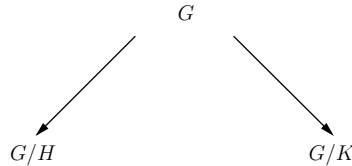
$$R^t \phi(x) = \int_{\hat{\xi}} \phi(\xi) d\xi.$$

Helgason then proved the following form of the formal adjoint theorem.

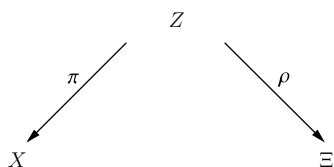
Theorem 5.73 (Helgason [270]). *If X and Ξ are homogeneous spaces in duality, then for any $f \in C_0(X)$, $\phi \in C_0(\Xi)$ we have*

$$\int_X f(x) R^t \phi(x) dx = \int_{\Xi} Rf(\xi) \phi(\xi) d\xi.$$

The situation so far can be visualized by the following diagram wherein the arrows are the quotient maps onto the respective homogeneous G spaces.

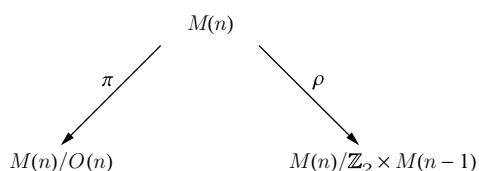


However, it is advantageous to modify the situation somewhat. If we replace the group G by a certain homogeneous space, then we lose nothing, but do gain some fiber bundles. To do this let $L = H \cap K$, as above, and define $\pi : G/L \rightarrow G/H$ by $\pi(gL) = gH$ and $\rho : G/L \rightarrow G/K$ by $\rho(gL) = gK$. These maps are well defined and continuous: to see that they are well defined, note that $g_1 L = g_2 L$ if and only if $g_2^{-1} g_1 \in L = H \cap K$. In particular, $g_2^{-1} g_1$ is in H so $g_1 H = g_2 H$. Therefore, π is well defined. A similar argument applies to ρ . Also a similar argument shows that the map $\tau : G/L \rightarrow X \times \Xi$ defined by $\tau(gL) = (gH, gK)$ embeds G/L as a subspace Z of $X \times \Xi$. In fact, Z consists of all (x, ξ) such that x is incident to ξ . With this identification of G/L with Z , the maps π and ρ become the coordinate maps: $\pi(x, \xi) = x$ and $\rho(x, \xi) = \xi$. Also $\pi^{-1}(x)$ consists of all (x, ξ) with ξ incident to x , that is, $\pi^{-1}(x) = \check{x}$. Since all the $\check{x}$ can be identified with H/L (by the corollary), it then follows easily that $\pi : Z \rightarrow X$ is a fiber bundle with fibers isomorphic to H/L . In the same way $\rho : Z \rightarrow \Xi$ is a fiber bundle with fibers isomorphic to K/L and with $\rho^{-1}(\xi) = \hat{\xi}$. This gives rise to the following diagram in which the maps are fibrations. For this reason we call the diagram a **double fibration**.



We now sketch a few examples of double fibrations which arise from homogeneous spaces in duality.

Example 5.74. The usual Radon transform on $\mathbb{R}^n$ can be represented as a double fibration. By remark 3.14, the manifold $\mathcal{G}_{n-1,n}$ of hyperplanes can be identified with $M(n)/\mathbb{Z}_2 \times M(n-1)$. Also $\mathbb{R}^n$ can be identified with $M(n)/O(n)$: consider the surjective map $M(n) \rightarrow \mathbb{R}^n$ defined by $g \mapsto g(0)$. The isotropy group of this mapping consists of all rigid motions such that $g(0) = 0$. But any such motion is an orthogonal transformation. Therefore, the isotropy group is $O(n)$ and $\mathbb{R}^n = M(n)/O(n)$. Therefore the usual Radon transform on $\mathbb{R}^n$ is an example of homogeneous spaces in duality defined by the following double fibration.



We just mention a few other examples for which Radon transforms can be defined via the Helgason double fibration process. Consult Helgason [291] for details.

Example 5.75. The k -plane transform can be derived via homogeneous spaces in duality by using a generalization of the case $k = n - 1$ just mentioned.

Example 5.76. The Funk transform. This generalized Radon transform integrates a function on the 2-sphere over the family of great circles. Unlike the ordinary Radon transform, a function on S^2 is not uniquely determined by this Radon transform. However, even functions are uniquely determined and an exact inversion formula can be derived.

The group $SU(1,1)$ is defined as the set of complex matrices of the form

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

with determinant 1, under the operation of matrix multiplication. There are several interesting double fibrations that arise from homogeneous spaces of this group. In the following examples $G = SU(1,1)$ and we describe the subgroups giving rise to the

following double fibration:

$$\begin{array}{ccc} & G = SU(1,1) & \\ \swarrow & & \searrow \\ X = G/H & & \Xi = G/K \end{array}$$

The hyperbolic, non-Euclidean disk $\mathbb{H}^2$ is modeled by the subset $\{|z| < 1\} \subset \mathbb{R}^2$ with Riemannian structure given by

$$ds^2 = \frac{|dz|^2}{(1 - |z|^2)^2}. \quad (5.54)$$

The geodesics in this geometry are the circular arcs in $\{|z| < 1\}$, which are perpendicular to the boundary. Horocycles of $\mathbb{H}^2$ are circles contained in the disk and which are tangent to the boundary.

Example 5.77. It can be shown that $\mathbb{H}^2$ is of the form $SU(1, 1)/H$, where H is the subgroup of matrices of the form $\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$, $\theta \in \mathbb{R}$, whereas the set of geodesics Ξ in this geometry has the form $SU(1, 1)/K$, where K is the subgroup generated by products from the subgroups $K_1 = \{\pm I, \begin{bmatrix} \pm i & 0 \\ 0 & \mp i \end{bmatrix}\}$, $K_2 = \{\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} : t \in \mathbb{R}\}$. The associated generalized Radon transform corresponds to integrating over geodesics.

Example 5.78. Again we deal with $\mathbb{H}^2$, and this time let $\Xi = SU(1, 1)/K$, where $K = K_1 K_2$ with $K_1 = \{\pm I\}$, $K_2 = \{\begin{bmatrix} 1+it & -it \\ it & 1-it \end{bmatrix} : t \in \mathbb{R}\}$. It turns out that the horocycles of $\mathbb{H}^2$ can be identified with the homogeneous space Ξ . The associated generalized Radon transform corresponds to integration over horocycles.

Example 5.79. In this example $X = SU(1, 1)/H$, where H is generated by the product of the subgroups $\{\pm I\}$, $\{\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\}$, $\{\begin{bmatrix} 1+it & -it \\ it & 1-it \end{bmatrix}\}$, whereas $\Xi = \mathbb{H}^2$. The corresponding Radon transform is the same as the Poisson integral and is inverted as in classical complex analysis.

Example 5.80. Theta series and cusp forms can be generated by other double fibrations of homogeneous spaces of $SU(1, 1)$.

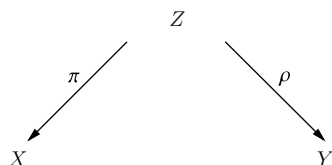
As pointed out in Helgason [287], the variety of examples illustrated by these generalized Radon transforms shows that the framework is too broad to yield useful individual results, although it is useful as a general tool.

Helgason was also the first to consider a Radon transform defined on Grassmannians. As an application of homogeneous spaces in duality Helgason [269] considers the case where X consists of p planes and Ξ consists of q planes in the Euclidean space $\mathbb{R}^n$ where $n = p + q + 1$. This is the Radon transform between Grassmannians. This transform goes from $\mathcal{G}_{p,n}$ to $\mathcal{G}_{q,n}$, where $n = p + q + 1$. A p plane in $\mathcal{G}_{p,n}$ is incident to a q plane in $\mathcal{G}_{q,n}$ if the two planes meet in a nonempty set. The Radon transform is then defined as follows: for $L \in \mathcal{G}_{q,n}$, $Rf(L)$ is defined to be the integral over all p planes which are incident to L , with respect to Haar measure on $\mathcal{G}_{p,n}$. Helgason

developed an inversion formula for this transform. However, his inversion formula required restrictions on the functions because of convergence problems arising when the p planes meet at small angles. Gonzalez [220] avoids these convergence difficulties by modifying the incidence relation to planes meeting orthogonally.

Here is a bit of the history and some references for the development of homogeneous spaces in duality. In 1963 Helgason developed the idea of the dual space of a symmetric space [266]. This was generalized to the idea of homogeneous spaces in duality which was developed by Helgason [267] and [269] and pursued in [268], [270], [271], [272], [276], [278]–[286], [288], [289], and [290]. A detailed development of Helgason's theory, including many of the results of these papers is presented in his books [292], [287], and [291]. The last reference is an especially good adjunct to the present volume. Also see Gel'fand, Graev, and Shapiro [193].

Gel'fand, Graev, and Shapiro [193] were able to generalize Helgason's Radon transforms generated by homogeneous spaces in duality. Gel'fand, Graev, and Shapiro dropped the requirement that the spaces involved in the double fibration be homogeneous spaces obtained from a topological group. They considered the situation of the double fibration in the following diagram:



In this diagram X and Y are C^∞ manifolds and Z is a C^∞ submanifold of $X \times Y$. The mappings $\pi : Z \rightarrow X$ and $\rho : Z \rightarrow Y$ define fiber bundles. Points $x \in X$ and $y \in Y$ are defined to be incident if $\exists a \in Z$ such that $\pi a = x$ and $\rho a = y$. Gel'fand, Graev, and Shapiro then define differential forms and their integrals over the fibers. This generates two Radon transforms, dual to each other (compare, Guillemin [245] and Guillemin and Sternberg [248]). The Gel'fand, Graev, and Shapiro double fibering is a generalization of Helgason's idea of homogeneous spaces in duality. Essentially, the requirement that the fiberings be generated by homogeneous spaces has been dropped.

Guillemin [243]³ refined the Gel'fand, Graev, and Shapiro double fibration by adding the conditions that the fiber mapping $\pi : Z \rightarrow X$ be proper (i.e., inverse images of compact sets are compact) and that the double fibration satisfy a certain microlocal injectivity condition called the Bolker assumption. Under these conditions Guillemin was able to show that if $\mathcal{R}$ and its dual $\mathcal{R}^\#$ are generalized Radon transforms generated by such a double fibration, then $\mathcal{R}$ and $\mathcal{R}^\#$ are Fourier integral operators and that $\mathcal{R}^\# \mathcal{R}$ is an elliptic pseudodifferential operator. From this it can be shown that $\mathcal{R}$ is locally invertible.

³ These results were presented by Guillemin at the Symposium on Global Analysis, held in Durham, England, July 1976. To my knowledge, no proceedings were ever published from that symposium. In 1984, Guillemin gave a talk at the conference on pseudodifferential operators and applications in which he finally presented this material see Guillemin [245]. Also see the book on geometric asymptotics by Guillemin and Sternberg [248].

The Bolker assumption for double fibrations derives from a result of Bolker showing that a Radon transform defined on a finite set is injective under a certain condition now known as the Bolker assumption. See section 5.10.5 for a precise statement of the Bolker assumption for Bolker's finite Radon transform.⁴ Referring to the double fibration above, the microlocal form of Bolker's assumption developed by Guillemin is: Let Γ be the conormal bundle $T^*(Z)$ with its zero section removed. Bolker's assumption is that $\rho : \Gamma \rightarrow T^*Y$ is injective. Let us call a **double fibration of the Guillemin type** one in which the fiber bundle $\pi : Z \rightarrow X$ is proper and in which the Bolker assumption is satisfied.

Quinto [507] has a survey of microlocal techniques in tomography with applications to partial differential equations. Also Greenleaf and Uhlmann [235] have a nice survey of microlocal techniques in integral geometry which includes an introduction to Guillemin's work on double fibrations and the local injectivity of generalized Radon transforms satisfying the Bolker assumption.

There are severe topological restrictions on the types of generalized Radon transforms that can arise from double fibrations of the Guillemin type. Quinto [499] showed that if we have a double fibration of the Guillemin type in which $\dim X = \dim Y > 2$, then the codimension of Z in $X \times Y$ must be 1, 2, 4, or 8. This means that the codimension of the varieties that we integrate over to obtain the dual Radon transform are severely restricted by the Bolker assumption. On the other hand, the Bolker assumption appears to be necessary for the proof of local invertibility for this type of generalized Radon transform. Quinto [499] also has some other interesting topological results: if $G_x = \pi^{-1}(x)$ is the fiber over $x \in X$, if k is the codimension of Z in $X \times Y$, and $n = \dim X$, then $k = 1$ implies G_x is diffeomorphic to S^{n-1} or $\mathbb{R}P^{n-1}$, if $k = 2$, then n must be even and G_x is homotopy equivalent to $\mathbb{C}P^{\frac{n}{2}-1}$ and if $k = 8$, then n must equal 16 and G_x is homeomorphic to S^8 . See Quinto [499] for further results. Also see Quinto [498].

The hyperbolic disk $\mathbb{H}^2$ can be generalized to real hyperbolic n space $\mathbb{H}^n$ by generalizing the metric defined in (5.54) to the unit ball in $\mathbb{R}^n$. See Helgason [265] or [292] (Berenstein and Casadio Tarabusi [44] have a brief introduction to this idea.) Helgason [265] gave inversion formulas for a k -plane transform defined on $\mathbb{H}^n$ for even k . Helgason [283] later generalized this to an inversion formula for any k . Let P be the k -plane transform on $\mathbb{H}^n$. Berenstein and Casadio Tarabusi [43] give an alternate inversion formula for $P^\#P$ on $\mathbb{H}^n$, for odd k . This inversion formula is the composition of a convolution operator with $p(\Delta)$, where Δ is the Laplacian and p is an explicit polynomial. Berenstein and Casadio Tarabusi [44] give yet another inversion of the k -plane transform on real hyperbolic space with the exception of a few dimensions. This formula involves a generalization of the Riesz potential to hyperbolic space. In this paper they also have a characterization of the range of the k -plane transform on real

⁴ Bolker's assumption and theorem has been known at least since 1980. References to his results appear in Quinto [498] and [499] and Guillemin [245]. The first published result of Bolker's theorem for the finite Radon transform that I can find is in Bolker [57].

hyperbolic spaces in terms of a system of second-order partial differential equations or as a single fourth-order equation (compare, Ishikawa [318]).

Impedance tomography is the process of applying a current to the boundary of an object, observing the resulting potential on the boundary and trying to determine the unknown conductivity of the interior of the object. The application of the current and the observation of the boundary potential is analogous to establishing an x-ray source and an x-ray detector in computerized tomography (CT). The unknown interior attenuation coefficient of CT is analogous to the unknown interior conductivity in impedance tomography. Barber and Brown [35] found, in a heuristic way, an approximate solution of a simplified version of impedance tomography. Santosa and Vogelius [544] showed that the problem of Barber and Brown could be handled by a technique for generalized Radon transforms introduced by Beylkin [54] (this is discussed in section 5.10.3). But Beylkin's technique only gives the inversion modulo a compact operator. Berenstein and Casadio Tarabusi [43, 46] observed that the simplified impedance tomography of Barber and Brown could be modeled by the x-ray, transform on the real hyperbolic plane. Berenstein and Casadio Tarabusi's inversion formula for the k -plane transform, $k = 1$, then gives an inversion formula for this simple case of impedance tomography. Also see Fridman et al. [181], Fridman et al. [180], Lissianoi and Ponomarev [392], and Berenstein [41].

The application of the Radon transform on hyperbolic spaces to impedance tomography works only in the simplified, linearization case of Barber and Brown. The general case of impedance tomography is difficult and its study is in its infancy.

5.6 The Radon Transform, Twistor Theory, and the Penrose Transform

In this section we require the use of certain projective spaces. Consider $\mathbb{C}^{n+1} \setminus \{0\}$ with an equivalence relation defined in the following way: for $z, w \in \mathbb{C}^{n+1} \setminus \{0\}$, then $z \sim w$ if and only if there is a nonzero complex number α such that $z = \alpha w$. Then we define $\mathbb{CP}^n$ to be the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ modulo this equivalence relation. This space $\mathbb{CP}^n$ is a compact, complex n -dimensional complex manifold and is called **complex projective space** of dimension n . The usual coordinates in $\mathbb{C}^{n+1} \setminus \{0\}$, with one nonzero coordinate fixed, act as local coordinates in $\mathbb{CP}^n$. These are called **homogeneous coordinates**. Each equivalence class in $\mathbb{CP}^n$ is a one-dimensional complex vector subspace of $\mathbb{C}^{n+1}$ with the origin missing. Therefore, there is a one-to-one correspondence between points of $\mathbb{CP}^n$ and one-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. If we define complex Grassmannians $\mathbb{CG}_{k,n}$ in an analogous manner to the real Grassmannians defined in chapter 3, then $\mathbb{CP}^n$ would equal $\mathbb{CG}_{1,n}$. Finally, $\mathbb{CP}^1$ is biholomorphic to the Riemann sphere S^2 in the sense that there is a one-to-one holomorphic mapping with holomorphic inverse between these complex manifolds. Any complex submanifold of $\mathbb{CP}^n$ that is biholomorphic to the Riemann sphere is called a **complex projective line** or simply a complex line.

Take $\mathbb{R}^4$ with the Lorentz metric $ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$ (we call this a metric with signature $(+ - - -)$). When we compactify this space it is called Minkowski

space M and is a natural domain for considering certain physical problems. For example, in special relativity, light travels on paths with $ds^2 = 0$. Such paths are called null lines. The null lines in M are precisely the paths of zero rest mass particles such as photons.

Let $\mathbb{M}$ be the set of complex projective lines in $\mathbb{CP}^3$. It turns out that $\mathbb{M}$ is a real four-dimensional complex manifold which is the complexification of a compactification of Minkowski space M .

Define twistor space $\mathbb{T}$ to be $\mathbb{C}^4$ with the Hermitian form Φ of signature $(+ + - -)$. Twistors are defined to be elements of $\mathbb{T}$. Positive twistors are those with $\Phi > 0$, null twistors are those with $\Phi = 0$, and negative twistors are those with $\Phi < 0$. It is useful to projectivize twistor space by using the Cartesian coordinates of a twistor as homogeneous coordinates in projective space. Therefore, projectivized twistors are precisely the elements of $\mathbb{CP}^3$ that we denote by $\mathbb{PT}$ when the form Φ is taken over to $\mathbb{CP}^3$ by computing in homogeneous coordinates. Let $\mathbb{PT}^+$, $\mathbb{PT}^0$, and $\mathbb{PT}^-$ be the positive, null, and negative projectivized twistors.

Twistors were introduced by Penrose [471] in 1967 to study the 15-parameter group of conformal transformations leaving the light-cone of Minkowski space invariant. Penrose then showed that twistor theory provided an important reformulation of relativistic physics and offered insights into the nature of space-time, quantum field theory, and elementary particles. See Wells [619] for further references to Penrose's work.

The Penrose correspondence gives a one-to-one mapping between complex lines in $\mathbb{PT}^0$ and points of Minkowski space M . It also gives a dual correspondence between points in $\mathbb{PT}^0$ and null lines in M . This duality is reminiscent of homogeneous spaces in duality and indeed can be represented by a double fibration.

The Penrose transform is analogous to a generalized Radon transform. The idea of the Penrose transform is to associate a function ϕ defined on Minkowski space to certain rational functions defined on twistor space. We do this in the projectivized spaces. Therefore a point x in Minkowski space, is represented by a point in $\mathbb{M}$ which is a projective line in $\mathbb{CP}^3$, also denoted by x . Therefore, we can integrate a rational function f over this projective line x and obtain a complex number. If we do this for all x , then we obtain a complex valued function ϕ which is called the Penrose transform of f . An additional requirement is that f be homogeneous of degree -2 . Then the Penrose transform takes the form

$$\phi(x) = \int_x f$$

where $x \in \mathbb{M}$. Since f is defined on $\mathbb{CP}^3$ which we identify with $\mathbb{PT}$, then we see that the Penrose transform is obtained by integrating functions on twistor space over projective lines in that space. This is clearly analogous to the Radon transform.

Penrose proved that the Penrose transform ϕ associated to a rational function f on twistors is a solution to the four-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x_0^2} - \frac{\partial^2 \phi}{\partial x_1^2} - \frac{\partial^2 \phi}{\partial x_2^2} - \frac{\partial^2 \phi}{\partial x_3^2} = 0$$

on M . Actually Penrose rediscovered this fact which was originally proved by Bateman [39]. However, Penrose did extend his transform to the case where f is homogeneous of degree $-2s - 2$. This idea led to solutions of massless Dirac equations of spin s . In specific cases these give: for $s = \frac{1}{2}$ solutions of the Dirac–Weyl equation for a neutrino, for $s = 1$ solutions of Maxwell’s equations and for $s = 2$ solutions of the linearized Einstein field equations of general relativity.

Two different rational functions could lead to the same solution. This ambiguity is resolved by treating the rational functions as Čech cocycles in a certain sheaf. Therefore, the Penrose version of the Radon transform generalizes to act on cohomology classes.

The Penrose correspondence and the Penrose transform can be formalized in the context of generalized Radon transforms arising from double fibrations. For details on this approach, see Wells [620].

Further references on twistors and their relation to Radon transforms may be found in Henkin and Polyakov [293], Polyakov and Henkin [495], Eastwood, Penrose, and Wells [140], and Eastwood [139].

5.7 The Radon Transform and $\bar{\partial}$ Cohomology

This section requires some knowledge of sheaf cohomology theory and several complex variables. Gindikin’s survey article “Between Integral Geometry and Twistors” [201] is a good introduction to the ideas of this section.

Let D be a linearly concave domain in $\mathbb{C}^n$, which is, by definition, a domain equal to the union of the complex hyperplanes contained in it. Let $\tilde{D} = \mathbb{C}^n \setminus \bar{D}$. Such a domain $\tilde{D}$ is called linearly convex and is a domain of holomorphy in $\mathbb{C}^n$, that is, not every holomorphic function on $\tilde{D}$ extends to a larger domain.

We let the dual domain D^* of D be the complex manifold consisting of complex hyperplanes contained in D . For any domain D in $\mathbb{C}^n$ an analytic functional on D is defined to be a continuous linear functional on the Frechet space $H(D)$ of all holomorphic functions on D .

Then it is known that if D is a linearly concave domain with a sufficiently regular boundary, then there is a one-to-one correspondence between analytic functionals on $\tilde{D}$ and holomorphic functions on D^* .

In addition to this representation of analytic functionals on $\tilde{D}$ as holomorphic functions on D^* there is a representation of analytic functionals as $(n - 1)$ -dimensional $\bar{\partial}$ -cohomology classes with coefficients in Ω^n . Gindikin and Henkin [206, 207] showed that the connection between these representations of analytic functionals is given by a type of generalized Radon transform which integrates $\bar{\partial}$ -cohomology classes over the hyperplanes in D^* .

More generally, an $(n - 1 - q)$ concave domain is one which is a union of the q -dimensional complex planes contained in the domain. Then (0) concavity is the same as linear concavity. The dual domain D^* of an $(n - 1 - q)$ concave domain D is the complex manifold whose points are the complex q planes contained in D .

Gindikin and Henkin proved that there is a differential operator κ and a generalized Radon transform R such that for any $(n - q - 1)$ -linearly concave domain $D \subset \mathbb{CP}^n$

such that D^* is contractible, there exists an associated holomorphic bundle E over D such that κR maps (n, q) forms on D into $(n + q)$ forms on E such that the restriction of $\kappa R\varphi$ to every section of E over D is cohomologous to φ . In this sense κ acts as an inverse to the generalized Radon transform R , because it is possible to construct the cohomology class of the (n, q) form φ from the cohomology class of the $(n + q)$ form which is its Radon transform.

The range of this generalized Radon transform is characterized by a system of differential equations in analogy to the case of the Radon–John transform when $k < n - 1$. In the case of the Gindikin–Henkin transform, if we define $k = n - 1 - q$, then the range of this transform when $k < n - 1$ (i.e., $q > 0$) consists of $(n + q)$ forms which satisfy a system of constant coefficient partial differential equations.

Details for these results may be found in Gindikin and Henkin [206, 207] and Gindikin [199].

As an application of their ideas, Gindikin and Henkin were able to formulate the Penrose correspondence and the Penrose transform in the context of their generalized Radon transform on $(n - 1 - q)$ concave domains: see Gindikin and Henkin [208, 209] and Gindikin [200].

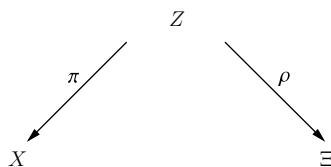
Another connection between cohomology and the Radon transform is described by Gindikin [204]. The space of Sato hyperfunctions on $\mathbb{R}^n$ is defined to be the $(n - 1)$ order $\bar{\partial}$ -cohomology group $H^{(n-1)}(\mathbb{C}^n \setminus \mathbb{R}^n, \mathcal{O})$ where $\mathcal{O}$ is the sheaf of germs of holomorphic functions on $\mathbb{C}^n$. It would be interesting to know if one could embed usual functions in the space of hyperfunctions. Gindikin shows that this can be done for rapidly decreasing functions, and some other classes too, by constructing a cohomology class for a given function f by taking the convolution of the Radon transform $Rf(\theta, p)$ with the distribution $(p - i0)^{-n}$ and multiplying by a certain differential form. He also shows that when the resulting form is integrated it acts exactly like the inversion formula for the Radon transform and yields the function f . Also see Gindikin [199] which gives a construction for integral formulas for cohomology classes in $H^q(D, \mathcal{O}(-s))$, where D is an $(n - q)$ – linear domain in the n -dimensional complex projective space $\mathbb{CP}^n$ and $1 \leq s \leq n$. Gindikin also investigates a Radon transform associated to such cohomology classes. Further articles of interest are Gindikin [205], Henkin and Polyakov [293], and Ofman [456].

5.8 $\mathcal{D}$ Modules

This section requires some knowledge of sheaf theory.

Given a differentiable manifold X , a $\mathcal{D}_X$ module is a sheaf of modules over the sheaf of rings of germs of differential operators on X . This concept was introduced by Sato, Kawai, and Kashiwara (see Kashiwara [332] for references). One can see from the definition that $\mathcal{D}$ modules form a very general framework for the study of partial differential equations. Because of the connection between the Radon transform and partial differential equations, it is not surprising that there is a connection between $\mathcal{D}$ modules and the Radon transform.

D’Agnolo and Schapira [117] consider the following double fibration of complex analytic spaces where Z is a closed analytic subspace of $X \times Y$.



Among other things, D'Agnolo and Schapira develop a correspondence between the category of coherent $\mathcal{D}_X$ modules and the category of $\mathcal{D}_Y$ modules which are obtained as direct images under ρ of inverse images under π of $\mathcal{D}_X$ modules (all modulo certain connections and regularity assumptions). This gives rise to a type of Radon transform. As an application, the authors apply their general machinery to the Penrose correspondence and transform.

The theory of $\mathcal{D}$ modules uses Radon transform techniques and also serves as an organizing framework that unifies results about various species of Radon transforms. Further references about these ideas are: D'Agnolo and Schapira [114]–[118], D'Agnolo and Eastwood [113], D'Agnolo [111, 112], Goncharov [216], and Sparling [579].

5.9 The Finite Radon Transform

In computed tomography one can only deal with finitely many x-ray projections at one time (compare, chapter 1). This leads to the following question: how much information about an unknown function can be determined by projections from a finite number of directions?

If two functions have the same x-ray projections from a finite set of directions, then the x-ray projections of their difference is zero in this set of directions. Taking a finite number of projections yields a linear operator which we call the **finite x-ray transform**. Therefore, the structure of the kernel, or null space, of the finite x-ray transform is of great interest. We also define and investigate the **finite Radon transform** and its kernel.

The finite Radon transform is finite only in the sense that the number of directions is finite. For each such direction θ one computes $Rf(\theta, p)$ for the infinite number of $p \in \mathbb{R}$. There is another type of Radon transform, also called the finite Radon transform, that acts on spaces with a finite number of elements. Unless otherwise stated in this section, we take the term finite Radon transform to mean the Radon transform from finitely many directions.

The formal definition of the finite Radon transform depends on the concept of the Radon projection $R_\theta f$ of the function f , orthogonal to the direction $\theta \in S^{n-1}$ which was defined in definition 2.28. Likewise we need to recall the definition of the x-ray projection which may be found in definition 3.16.

Definition 5.81. Given a finite set of directions $\vartheta = \{\theta_1, \dots, \theta_p\} \subset S^{n-1}$ we define the **finite Radon transform** R_ϑ in the following way. Let f be a function defined on $\mathbb{R}^n$. Then we let

$$R_\vartheta f = (R_{\theta_1} f, \dots, R_{\theta_p} f).$$

The *finite x-ray transform* is defined by

$$X_{\vartheta} f = (X_{\theta_1} f, \dots, X_{\theta_p} f).$$

Remark 5.82. If f is restricted to a particular linear function space V on $\mathbb{R}^n$ with the property that the Radon projection $R_{\theta} f$ exists and lies in the function space W on $\mathbb{R}$, for each $f \in V$ and $\theta \in S^{n-1}$, then the finite Radon transform is a linear operator: $R_{\vartheta} : V \longrightarrow \bigoplus_{j=1}^p W_j$.

Remark 5.83. Recall that the x-ray projection $X_{\theta} f$ is a function defined on the orthogonal complement of θ . To define the range of the finite x-ray transform, then we need to specify, for each $j = 1, \dots, p$, a space of functions W_j on $\theta_j^{\perp}$. Then the finite x-ray transform is a linear operator: $R_{\vartheta} : V \longrightarrow \bigoplus_{j=1}^p W_j$.

Definition 5.81 provides the context for studying the problem posed above. However, the problem is still idealized because we assume that we have complete data for any single projection.

In chapter 3 we encountered the following results by Smith, Solmon, and Wagner [566]:

- An object is determined by any infinite set of radiographs (i.e., x-ray projections).
- For almost any finite dimensional space of objects F , the objects in F can be distinguished by a single radiograph from almost any direction.
- In general, a finite set of radiographs tells nothing at all.

The reader is referred to chapter 3, theorems 3.144, 3.148, 3.138 for the precise statements, proofs and comments about these results.

The latter two of the Smith, Solmon, and Wagner results relate directly to the finite x-ray and Radon transforms. Depending on which function space V we use, the finite x-ray or Radon transform may or may not uniquely determine a function. If V is finite dimensional, then we may expect the x-ray transform to be injective. But if V is infinite dimensional, then it is very unlikely for this to be true.

However, even if we cannot exactly determine a function f from finitely many projections, we can still ask how likely it would be to encounter a function radically different than f . For this purpose it is sufficient to understand the structure of the null space of these transforms. This can be done by examining the structure of the Fourier transform of functions in the null space. The Fourier transform approach leads to a category of theorems called “uncertainty principles”.

5.9.1 Uncertainty Principles for Radon Transforms: The Kernel of the Finite Radon Transform

It is interesting to investigate properties of the kernels of the operators X_{ϑ} and R_{ϑ} . One way of investigating these kernels is to look at the frequency distributions of their functions. Uncertainty principles for the Radon transform are concerned with the

frequency distribution of functions in the null space of the finite Radon transform. The first such uncertainty principle is from B. F. Logan [393] who developed this principle in $\mathbb{R}^2$. The next definition formalizes the idea of this frequency distribution.

Definition 5.84. For $f \in L^2(\mathbb{R}^n)$, $f \neq 0$, define

$$\lambda(f; c) = \frac{\|\hat{f}\|_{L^2(B(c))}^2}{\|\hat{f}\|_{L^2(\mathbb{R}^n)}^2} \quad (5.55)$$

We call $\lambda(f; c)$ the *frequency distribution of the function f in bandwidth c* .

As usual $\hat{f}$ denotes the Fourier transform of f and $B(c)$ denotes the ball of radius c centered at the origin.

In electrical engineering $\|\hat{f}\|_{L^2(B(c))}^2$ is a measure of the amount of energy of the function f in bandwidth c . Electrical engineers use the term bandwidth as a synonym for the radius of a ball centered at the origin. We adopt this terminology, so if we say that f is a high-bandwidth function, then we mean that most of the energy of f lies outside a ball of large radius (i.e., high bandwidth). This last case means that there is a large enough bandwidth c such that $\lambda(f; c)$ is close to zero. Therefore $\lambda(f; c)$ is a measure of how much of the energy of f is contained in bandwidth c . Conversely, if $\lambda(f; c)$ is close to 1, then most of the energy of f lies within bandwidth c .

Frequently one wants to filter out the high frequencies of a function. This is done by using a low-bandpass filter (relative to a bandwidth c). Such a filter Ψ_c has the property that multiplying the Fourier transform of a function f by $\hat{\Psi}_c$ results in annihilating most of the frequencies of f above bandwidth c . In other words, this filter only lets the lower frequencies pass through the filtering process. We can design a low-bandpass filter by constructing an L^2 function ψ_c , which is close to zero when $\xi > c$. If we then define $\Psi_c = \mathcal{F}^{-1}(\psi_c)$, then by the convolution theorem for the Fourier transform we get $\hat{f} * \hat{\Psi}_c(\xi) = (2\pi)^{\frac{n}{2}} \hat{f}(\xi) \psi_c(\xi)$. Since ψ_c is close to zero when $\xi > c$, then the convolution $f * \Psi_c$ is a function for which the frequencies of f higher than c have been filtered out. We call $f * \Psi_c$ the function obtained from f by filtering with the low-bandpass filter Ψ_c .

The ideal low-bandpass filter Φ_c filters out all the frequencies above bandwidth c . Hence, the ideal low band pass filter is the inverse Fourier transform of the characteristic function of the ball $B(c)$.

Definition 5.85. Define

$$\lambda_p(c) = \sup \left\{ \lambda(f; c) : f \in L^2(B(1)), f \neq 0 \right. \\ \left. \text{and } X_{\vartheta} f = 0, \text{ where } \vartheta \subset S^{n-1} \text{ is some set of } p \text{ directions} \right\}$$

The following theorem is from Logan [393] in dimension $n = 2$ and from Maass [407] for general n .

Theorem 5.86 (Logan–Maass uncertainty principle for the x-ray transform). *In $\mathbb{R}^n$ we have the following results. If $\gamma > 0$ and $\beta > \frac{1}{3}$, then*

$$\lim_{p \rightarrow \infty} \lambda_p \left(\left(p + \frac{n}{2} - 1 \right) - \gamma \left(p + \frac{n}{2} - 1 \right)^\beta \right) = 0 \quad (5.56)$$

while if $\gamma > 0$ and $\beta > \frac{1}{2}$, then

$$\lim_{p \rightarrow \infty} \lambda_p \left(\left(p + \frac{n}{2} - 1 \right) + \gamma \left(p + \frac{n}{2} - 1 \right)^\beta \right) = 1. \quad (5.57)$$

In particular, if $0 < \alpha < 1$, then

$$\lim_{p \rightarrow \infty} \lambda_p \left(\alpha \left(p + \frac{n}{2} - 1 \right) \right) = 0 \quad (5.58)$$

while if $\alpha > 1$, then

$$\lim_{p \rightarrow \infty} \lambda_p \left(\alpha \left(p + \frac{n}{2} - 1 \right) \right) = 1. \quad (5.59)$$

Proof. The reader is referred to Logan [393] and Maass [407] for the proofs of the two limit equations (5.56) and (5.57). Although Maass's result applies to any dimension, Logan's proof is more elementary, but applies only to $n = 2$. The proof of the last two limits is an easy consequence of (5.56) and (5.57). Just take $\beta = 1$ and $\gamma = 1 - \alpha$ to get equation (5.58) from equation (5.56). A similar technique works for the last equation. ■

Theorem 5.86 can be interpreted as follows, although this interpretation is not very rigorous. We will make the ideas more precise later, see theorems 5.87 and 5.88. Equation (5.58) indicates that if the number of directions p is sufficiently large and X_ϑ is a finite x-ray transform on $\mathbb{R}^2$, where ϑ contains p directions, then most of the energy of a function in the kernel of X_ϑ lies outside a bandwidth which is somewhat smaller than $p + \frac{n}{2} - 1$. Therefore, if we use the data from the finite x-ray transform to construct a function g with the same p projections as f , then the difference between f and g would be a high-frequency function. If we applied an appropriate low-bandpass filter to g , then the result should be approximately equal to f , provided that f itself does not contain many high-frequency components. The bandwidth of the low-bandpass filter would only need to be $p + \frac{n}{2} - 1$ or greater.

On the other hand, if we used somewhat fewer projections than the bandwidth $p + \frac{n}{2} - 1$, then most of the energy of elements of the null space of X_ϑ would be contained in this bandwidth. Therefore, the difference between the original function f and the reconstruction g would be a highly oscillating function. In this case the reconstruction g would be substantially different than f . This is the case illustrated in chapter 3, section 3.12.

The Logan–Maass uncertainty principle has no restrictions on the distribution of the directions, except that the directions need to be distinct. Louis [398] has a generalization of the Logan–Maass uncertainty principle to the case of the Radon transform. This case requires greater restrictions on the directions when $n > 2$. Also the number of directions

p is replaced by the number m where $p = (\frac{m}{n} - 1)^{-2}$. Besides this and the fact that the Louis uncertainty principle applies to the Radon transform, the statement of the Louis uncertainty principle is identical with that of the Logan–Maass uncertainty principle. We refer the interested reader to Louis [398] for the details.

The following theorem is an application of these ideas. In this result Φ_c is the ideal low-bandpass filter defined earlier in this section.

Theorem 5.87. *For the x-ray transform, if ϑ is a finite set of p directions in $\mathbf{S}^{n-1}$, if f_0 and f_1 are in $L^2(B(1))$ and if $X_{\vartheta} f_0 = X_{\vartheta} f_1$, then*

$$\|f_0 * \Phi_c - f_1 * \Phi_c\| \leq \sqrt{\lambda_p(c)} \|f_0 - f_1\|.$$

This theorem is originally from Logan [393] in dimension $n = 2$. Higher-dimensional versions are from Maass [407] for the x-ray transform and from Louis [398] for the Radon transform. The norms appearing in this theorem are L^2 norms. Cheung and Markoe [89] have a version of this result where Sobolev norms are used. We refer the reader to these references for the proofs.

An immediate consequence of this theorem is that under the stated hypotheses, $\|f_0 * \Phi_c - f_1 * \Phi_c\| \leq 2\sqrt{\lambda_p(c)} \max(\|f_0\|, \|f_1\|)$. Since $\lambda_p(c)$ tends to zero very quickly once $p + \frac{n}{2} - 1 > c$, and since $f * \Phi_c$ filters out frequency components of f in bandwidths higher than c , then this result has the following implication: two functions of moderate L^2 norm, with negligible high-frequency components above bandwidth c , and with the same x-ray projections from more than c directions must be close to each other.

In reference to the size of $\lambda_p(c)$, in dimension $n = 2$ if one takes the bandwidth c to be 90% of the number of directions p , then $\lambda_p(c) < 3.5 \times 10^{-4}$, when $p = 20$ and $\lambda_p(c) < 6.1 \times 10^{-35}$, when $p = 180$.

Khalfin and Klebanov [349, 350] and Klebanov and Rachev [355] have similar results to theorem 5.87.

Theorem 5.87 applies to any set of p distinct directions in $\mathbb{R}^n$ but requires precise knowledge of the x-ray projections in these directions. The result works even if the directions are very close to each other because one knows precisely the values of these x-ray projections. Cheung and Markoe have the following generalization of theorem 5.87 which applies to functions with finitely many x-ray projections which are close but not necessarily identical. The concept of singular value decomposition is needed for this result. The reader is referred to Natterer [444] for the definition of singular value decompositions. The norms used in this theorem are weighted Sobolev norms. In the statement of this result H^s refers to a Sobolev space of order s and the norms are Sobolev norms. The reader is referred to Cheung and Markoe [89] for details about these norms as well as the proof.

Theorem 5.88 (Cheung and Markoe [89]). *Let p be a positive integer, let ϑ be a set of p directions in $\mathbf{S}^{n-1}$ and let $c > 0$. Let w be a weight function for which the finite x-ray transform $X_{\vartheta} : L^2(B(1)) \longrightarrow \bigoplus_{j=1}^p L^2([-1, 1], w)$ has a singular value decomposition with singular values σ_i . Letting $\sigma = \inf \sigma_i$, we then have for*

$f, g \in H^s(B(1))$, where $s > 0$

$$\|f * \Phi_c - g * \Phi_c\| \leq \frac{1}{\sigma} \|X_\vartheta f - X_\vartheta g\| + \sqrt{\lambda_p(c)} \|f - g\|$$

A similar result holds for the finite Radon transform.

There are weights w for which the minimum singular value σ is positive. However, as shown in Davison and Grünbaum [121] and Davison [119, 120], σ tends to be very small when the directions are close together. On the other hand, Davison and Grünbaum [121] showed that for certain weights and for p equally spaced angles, then $\sigma \geq \sqrt{\frac{1}{p}}$; this is a very mild degeneration of the singular values.

Let us consider two functions f and g of moderate Sobolev norm, with negligible high-frequency components above bandwidth c , and with x-ray projections from more than c directions that are spread out enough to counteract the factor $\frac{1}{\sigma}$. Then theorem 5.88 implies that f and g must be close to each other in the Sobolev norm.

Khalfin and Klebanov [350] and Klebanov and Rachev [354] have results similar to theorem 5.88.

The paper [121] of Davison and Grünbaum mentioned above also investigates the following variant of the filtered backprojection method. In standard filtered backprojection one chooses a single filter that is applicable to all directions. Davison and Grünbaum show how to design a filter for each direction θ of a given finite set of directions in such a way as to optimize the reconstruction process.

5.10 Additional References and Results

Ehrenpreis [149] is a booklength treatment of an alternative approach to the study of generalized Radon transforms. Ehrenpreis introduces the notion of a “spread” as a generalization of the idea of slicing a space X into families of manifolds similarly to the way that one slices $\mathbb{R}^n$ into families of hyperplanes in the case of the usual Radon transform or the way that a double fibration slices a space to create a generalized Radon transform. As a prerequisite to this book we suggest the articles [146], [147], and [148] by Ehrenpreis. Also, Kuchment and Quinto have written an appendix [364] to Ehrenpreis’s book that gives nice introduction to the state of the art of the mathematics used in tomography as of the beginning of the twenty-first century.

Louis has two interesting papers [400] and [401] both with the title “Medical Imaging: State of the Art and Future Development.” The first paper [400] is fairly elementary from the mathematical point of view but has interesting informal descriptions of the process of data collection and reconstruction of images for medical diagnosis. The second paper is a very detailed overview of both the process and the mathematics of various forms of medical tomography. It contains discussions of several of the topics treated in this book, including local tomography, the Radon transform, the attenuated Radon transform, the exponential Radon transform, cone beam tomography, the FDK algorithm, and Grangeat’s method. It also deals with magnetic resonance imaging, ultrasound tomography, diffuse tomography, impedance tomography, and limited-view

tomography. The emphasis in [401] is practical with attention being paid to discretization, stability, conditioning, and development of practical algorithms for tomography. For the reader interested in the practical side of tomography, the articles [400] and [401] also provide an extensive list of references dealing with applications to medical imaging.

Rubin [536] contains a survey of several topics considered in this chapter.

5.10.1 Divergent Beam and Cone Beam Transforms

Natterer [442] has some interesting ideas on various areas of tomography, in particular, on cone-beam reconstruction methods.

Palamodov [462] analyzes the error in cone-beam reconstruction methods.

Faber, Katsevich, and Ramm [154] study cone-beam tomography with an application to helical tomography.

Further references are Noo et al. [451], Defrise, Noo, and Kudo [126], Katsevich [341]–[345], and Palamodov [466].

5.10.2 Attenuated and Exponential Radon Transforms

The study and inversion of the attenuated Radon transform is closely related to the study of the transport equation. The earliest result in this direction was due to Bellini et al. [40] and its culmination is in the work of Arbuzov, Bukhgeim, and Kazantsev [24], who were the first to show that the attenuated Radon transform is invertible, and Novikov [453] who produced the first inversion formula for this transform. Anikonov, Kovtanyuk, and Prokhorov have a booklength treatment [19] of the relation of the transport equation to the attenuated Radon transform (compare, Gourion and Noll [226]).

Arbuzov, Bukhgeim, and Kazantsev [24] reduced the problem of invertibility to the study of a transport equation which could be solved by their theory of A analytic functions. The theory of A analytic functions is described in Bukhgeim [71]. Related references are Bukhgeim and Kazantsev [72], Bukhgeim [70], Arbuzov and Bukhgeim [21, 22], and Arbuzov, Bukhgeim, and Kazantsev [23].

Feig and Greenleaf [165] consider an integral transform related to the attenuated Radon transform. This transform, which is useful in radar detection problems, is defined by $F(D)(x, y) = \int_{-\infty}^{\infty} D(sx, s + y)e^{ixs^2} ds$ for functions D defined on $\mathbb{R}^2$. They show that given F , a linear transformation A of $\mathbb{R}^2$ exists such that $D + D \circ A$ can be reconstructed from F . In the special case that D is real valued and vanishing on the half-plane $x < 0$, they show that one can recover D . Feig and Grünbaum [166] also have a result on the relation between tomography and radar detection.

Heike [263] studies the attenuated Radon transform on $\mathbb{R}^2$ with C^∞ attenuation, obtaining an approximate inversion procedure. He proves that R_μ is a bounded operator between the Sobolev spaces $H^s(B(1))$ and $H^{s+\frac{1}{2}}(S^{n-1} \times \mathbb{R})$. This follows from some results of Quinto [498], but Heike gives an independent proof. He also proves that R_μ is a semi-Fredholm operator. These facts are used to derive an algorithm to approximately invert the attenuated Radon transform. He also proves by Sobolev space methods that R_μ is locally invertible. We note that this is a consequence of a result of Guillemin [243] who

proved local invertibility for certain generalized Radon transforms with C^∞ weights (compare, Guillemin [245] and the discussion in section 5.5). As shown in theorem 5.35, Markoe and Quinto [424] also proved local invertibility for generalized Radon transforms with C^2 weights.

Kunyansky [368] treats the reconstruction problem for the attenuated Radon transform in a novel way. His idea is to apply the usual Radon transform to the observed data for the attenuated Radon transform. Of course, the resulting reconstruction will not be correct. Kunyansky interprets the difference between the true image and the imperfect reconstruction as noise and suggests an algorithm to reduce the noise and thereby obtain a reasonable reconstruction for the attenuated Radon transform.

The earliest consistency conditions for the attenuated Radon transform were given by Natterer in [437], [438]. These conditions were only necessary and not sufficient. They were described in theorem 5.25, whereas the best source for a proof is in Natterer [444]. In [440] Natterer applies these ideas to the transport equation. Necessary and sufficient range and consistency conditions for the exponential Radon transform were developed by Kuchment and L'vin [362], [363], Aguilar and Kuchment [14], Aguilar, Ehrenpreis, and Kuchment [13]. Öktem [457] proves the range results of Aguilar, Ehrenpreis, and Kuchment [13] via a theorem on extending separately holomorphic functions of several complex variables. Kuchment [360] has both a range theorem and an inversion formula for a transform related to the exponential Radon transform. L'vin [406] uses range conditions for the exponential Radon transform to compensate for knowing data in only a limited segment of directions. The most general range result for these transforms is that of Novikov [454] who provided necessary and sufficient consistency conditions for a function to be in the range of the attenuated Radon transform.

Dicken [129] attempts to solve the identification problem for the exponential Radon transform via the use of Tikhonov regularization. Hertle [303] and Solmon [574] also have results on the identification problem for the exponential Radon transform.

The following result on generalized Radon transforms due to Boman is related to the identification problem for the exponential Radon transform discussed in section 5.3.

Theorem (Boman [60]). *Let $p > 1$. Let $\mu, \sigma \in C^\infty(\mathbb{R}^n)$ with $\mu, \sigma \geq 0$. If we are then given compactly supported measures f and g on $\mathbb{R}^n$ such that $\mathcal{R}_\mu f = \mathcal{R}_\sigma g$ and $f = f_0 + f_1$, where f_0 is a finite sum of Dirac measures and where $f_1 \in L_0^p(\mathbb{R}^n)$, then there is a function a such that $g = af_0 + g_1$, where $g_1 \in L_0^p(\mathbb{R}^n)$. Furthermore, for x in the support of f_0 , the function $\mu(x, \theta)/\sigma(x, \theta)$ is independent of the direction θ and for such x we have $a(x) = \frac{\mu(x, \theta)}{\sigma(x, \theta)}$.*

Boman also has a similar result for measures with positive dimensional support. He uses the theory of pseudodifferential operators to derive these results.

Rubin [532] generalizes the exponential Radon transform to an exponential k -plane transform and provides inversion formulas.

Solmon [572] shows that an L_0^1 function has a vanishing divergent beam x-ray transform from a finite source set if and only if a certain variable coefficient partial differential equation admits a solution with compact support. From this he shows that

two functions with the same divergent beam x-ray transform from a finite set of sources may differ arbitrarily on any compact set in the interior of their support.

5.10.3 The Generalized Radon Transform on Hyperplanes

Boman and Quinto [63] prove support theorems for real analytic generalized Radon transforms. Their main theorem is:

Theorem 2.1. [63] *Assume $(\omega_0, p_0) \in S^{n-1} \times \mathbb{R}$ and $f \in \mathcal{E}'(\mathbb{R}^n)$. Let $\mu(x, \omega)$ be a strictly positive, real analytic function on $\mathbb{R}^n \times S^{n-1}$ that is even in ω . Let V be an open neighborhood of ω_0 . Finally, assume $\mathcal{R}_\mu f(\omega, p) = 0$ for $\omega \in V$ and $p > p_0$. Then $f = 0$ on the half-space $\langle x, \omega_0 \rangle > p_0$.*

This theorem implies the following generalization of the classical support theorem for the classical Radon transform which was proved by Helgason [269] and Ludwig [405].

Corollary. *Under the same assumptions on weight function, if $\mathcal{R}_\mu f(\theta, p) = 0$ for all hyperplanes $\langle \theta, x \rangle = p$ missing a compact convex set $K \subset \mathbb{R}^n$, then the support of f is contained in K .*

See Boman [61] for support theorems of generalized Radon transforms on both $\mathbb{R}^n$ and projective $n -$ space. In this paper, the functions may satisfy a weaker condition than compact support (see also Takiguchi [593, 594]).⁵

Boman and Quinto also obtain the following generalization of the limited angle uniqueness theorem for the classical Radon transform (compare with theorem 3.142, chapter 3).

Theorem 2.3. [63] *Assume $J \subset S^{n-1}$ is contained in no proper real-analytic variety in S^{n-1} . Let $\mu(x, \omega)$ be a strictly positive, real analytic weight function. Let $f \in \mathcal{E}'(\mathbb{R}^n)$ satisfy $\mathcal{R}_\mu f(\omega, p) = 0$ for $\omega \in J$ and all real p . Then $f = 0$.*

Gonzalez [221] investigates the algebra of bi-invariant differential operators of the motion group of $\mathbb{R}^n$, and use these results to develop intertwining properties of generalized Radon transforms on $\mathbb{R}^n$ and on Grassmann manifolds.

Graev [227] uses a generalized Radon transform involving integration over sections of certain line bundles over $\mathbb{RP}^3$ to construct projection operators for the discrete and continuous series of the regular representation of the group $PSL(2, \mathbb{R})$.

Beylkin [54] has a generalization of the computation of the symbol of $\mathcal{R}^\# \mathcal{R}$ in Quinto [498] (see theorem 5.47). Beylkin defines an operator $\mathcal{R}_\mu^*$, related to the formal adjoint $\mathcal{R}_\mu^\#$, and he defines a pseudodifferential operator K such that $\mathcal{R}_\mu^* K \mathcal{R}_\mu$ is an elliptic pseudodifferential operator. He then expands $\mathcal{R}_\mu^* K \mathcal{R}_\mu$ into the identity operator plus a series $\sum T_j$ of operators, where T_j is a pseudodifferential operator of order $-j$.

⁵ In these papers Takiguchi claimed that there was an error in Boman [61]. However, Takiguchi only misinterpreted a statement in Boman's paper and that paper seems to be correct. See the review by B. Rubin: Math Reviews 2001m:44005 and also the summary 2002k:44003.

He is able to explicitly calculate T_1 . Using the results of [54], Beylkin [55] is able to find an approximate solution to a linearized inverse scattering problem which has applications to seismic data processing.

5.10.4 Generalized Radon Transforms on Other Spaces

The books Helgason [291], Gel'fand, Gindikin, and Graev [189], and Ehrenpreis [149] are good general sources for the material on homogeneous spaces in duality. The short paper [202] by Gindikin traces the heritage of Radon's 1917 paper to modern day mathematics, in particular, to integral geometry on more general spaces. A good survey on the aspects of integral geometry relative to double fibrations and homogeneous spaces in duality can be found in Guillemin [246]. Guillemin and Sternberg [249] is a good reference for the microlocal (Fourier integral operator and pseudodifferential operator) aspects of double fibrations.

The totally geodesic k -dimensional Radon transform integrates functions over totally geodesic dimension k submanifolds of a constant curvature Riemannian manifold. Helgason [265] studies a Radon transform on the classical non-Euclidean spaces of constant curvature in which the integration is performed over totally geodesic submanifolds (compare, Helgason [283]). Helgason [273] discusses harmonic analysis and the Radon transform on the Lobachevski plane. Gindikin [197] develops a Radon transform on the torus in which integration is over closed totally geodesic hyperplanes.

An equality due to Sobolev states that $\|f\|_{L^q(\mathbb{R}^n)} \leq c(\|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)})$ provided that $1 \leq p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$. Strichartz [586] comments that simple examples show that one cannot increase the index q if one remains in the full space $L^q(\mathbb{R}^n)$. He shows, however, that if one restricts f from $L^q(\mathbb{R}^n)$ to smaller classes, then it is possible to increase the index q in the Sobolev inequality. He has various applications of this result including the following one. He shows that the k -plane transform P_k on the sphere S^n , with its antipodal points identified, can be identified with an operator from $L^2(G_{1,n+1}) \rightarrow L^2(G_{k+1,n+1})$. The integrations for this k -plane transform are performed over the intersections of $k+1$ planes in $\mathbb{R}^{n+1}$ with the sphere S^n . These intersections form the totally geodesic submanifolds of S^n of dimension k . He then proves the following theorem: P_k is a bounded linear operator: $L^p(G_{1,n+1}) \rightarrow L^q(G_{k+1,n+1})$ provided the point $(\frac{1}{p}, \frac{1}{q})$ lies in the convex hull of the points $(0, 0)$, $(1, 1)$, $(\frac{1}{2}, \frac{1}{2} - \frac{k}{2n})$, $(\frac{1}{2} + \frac{k}{2n}, \frac{1}{2})$. This is an improvement on a similar result in his paper [584]. The only difference is that in the earlier paper the third point describing the convex hull had the value $(\frac{1}{2}, \frac{1}{2} - \frac{k}{2(n-k)(k+1)})$ which gives a smaller set unless $k = n - 1$. Kurusa [373] investigates the totally geodesic k -dimensional Radon transform on complete, simply connected Riemannian manifolds. Berenstein and Rubin [48] find explicit formulas for the inversion of both L^p functions and continuous functions on $\mathbb{H}^n$, (compare, Bray and Rubin [67], Rubin [531], and Berenstein and Rubin [49]).

Additional papers on totally geodesic Radon transforms are Berenstein, Casadio Tarabusi, and Kurusa [47], Palamodov [464], and Rubin [533]. Rubin [534] has inversion results for a generalization of the totally geodesic Radon transform, and Rubin [535] has inversion results for a generalization of the totally geodesic k -plane transform.

Further results on the Radon transform on symmetric spaces, beyond those cited in section 5.5 may be found in Eguchi [141]–[144].

Semjanistyi studies Radon transforms on projective spaces and other non-Euclidean spaces [550], [551] and [552] (with Petrov and Sibasov).

Gel'fand and Gindikin [190] investigate generalized Radon transforms on $\mathbb{R}^n$ and $\mathbb{C}^n$.

Orloff [458, 459] deals with the injectivity of invariant generalized Radon transforms on rank 1 symmetric spaces of the noncompact type.

Sibasov [560] studies a generalized Radon transform in spaces of matrices over the reals, complexes, and quaternions.

Hertle [297] generalizes the Radon transform to separable Banach spaces.

Zalcman [625] investigates some topics in integral geometry such as using integrals over families of submanifolds to determine properties of functions on a manifold. This is a generalization of the idea of the Radon transform.

Fritz and Oppel [182] develop generalized Radon transforms in an even more general setting in which we have measure spaces X and Y generalizing the duality between $\mathbb{R}^n$ and $G_{k,n}$ in the k -plane transform.

Cormack [103, 105] considers a generalized Radon transform on $\mathbb{R}^2$ which integrates functions over certain families of curves rather than lines. The first family of curves that he considers are called α -curves and are defined in polar coordinates by the equation

$$r^\alpha \cos(\alpha(\theta - \phi)) = p^\alpha$$

The variables θ and p parametrize the curves, just as in the case of the Radon transform. It is assumed that $\alpha > 0$, and that $|\theta - \phi| < \frac{\pi}{2\alpha}$. He also defines β -curves by

$$p^\beta \cos(\beta(\theta - \phi)) = r^\beta$$

with $\beta > 0$ and $|\theta - \phi| < \frac{\pi}{2\beta}$. For various choices of α and β one finds among these curves cardioids, circles through the origin, lemniscates of Bernoulli, orbits of charged particles moving in certain magnetic fields, and orbits of particles moving under a central force. Cormack shows how to invert these types of generalized Radon transforms and, in the second paper he examines the null spaces of the Fourier components of these transforms. Cormack [106] also has a generalization to $\mathbb{R}^n$ where he considers a family of Radon transforms $R^{(\alpha)}$ for certain real numbers α . This Radon transform is obtained by integrating rapidly decreasing functions over surfaces in $\mathbb{R}^n$ defined by

$$S^{(\alpha)}(\varphi, p) = \left\{ r\psi : r \in \mathbb{R}^+, \psi \in S^{n-1}, \langle \psi, \varphi \rangle \in \left(\cos\left(\frac{\pi}{2\alpha}\right), 1 \right] \right. \\ \left. \text{and } r^\alpha \cos\left(\alpha \cos^{-1}(\langle \psi, \varphi \rangle)\right) = p^\alpha \right\}.$$

If $\alpha = 1$, then $S^{(\alpha)}$ form hyperplanes and we have the classical Radon transform. If $\alpha = -1$, then $S^{(\alpha)}$ are circles and we have the spherical mean value transform. In these two cases inversion formulas were already known. Cormack gives an inversion formula, presents a hole theorem, and also determines the null space and gives consistency conditions for the Radon transforms $R^{(\alpha)}$ with $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$. Apparently Kulessa [367] has inversion formulas for a more general class of Radon transforms including Cormack's $R^{(\alpha)}$ for any α (compare, Romanov [523]).

Phong [485] showed how the Green's function for the $\bar{\partial}$ -Neumann problem can be written as a combination of a Hilbert integral operator and a certain generalized Radon transform.

Phong and Stein [487] introduced a generalization S of the Radon transform which is defined in the following way: assign a hypersurface to each point in $\mathbb{R}^n$, $n \geq 3$. The hypersurface must satisfy a certain curvature requirement, the simplest being nonvanishing Gaussian curvature. Integration is replaced by assigning a distribution to each such hypersurface and evaluating the restriction of a test function on $\mathbb{R}^n$ to each hypersurface. They prove that S is a bounded operator on L^p for $1 < p < \infty$ (compare, Phong and Stein [492, 493], Stein [582], and Lee [387]). In the special case that $Sf(t, x) = \int_{\mathbb{R}} f(t + Q(x, y), y)\psi(t, x, y)dy$, where Q is a homogeneous polynomial of degree n and where ψ is a cutoff function, Phong and Stein [491] showed that a necessary condition for S to be a bounded operator between L^p and L^q is that $(\frac{1}{p}, \frac{1}{q})$ be contained in a certain trapezoid in $\mathbb{R}^2$. In dimensions $n = 2, 3$ it was also shown that this condition is sufficient (Phong and Stein [491], Seeger [547, 548]). Bak [31] proved the sufficiency in the case $n \geq 4$.

Phong and Stein [488, 489] introduce the idea of singular Radon transforms to study Hilbert integrals and L^p estimates for solutions of the $\bar{\partial}$ -Neumann problem on $\mathbb{R}^n$, $n \geq 2$. In the first paper they consider the following situation. Let Ω be a C^∞ manifold and let Ω_x be a family of hypersurfaces in Ω smoothly varying with $x \in \Omega$. The authors assume that the hypersurfaces Ω_x have nondegenerate rotational curvature. Given such a family of hypersurfaces, the authors define a singular Radon transform to be an operator $T : C_0^\infty \rightarrow C^\infty$ defined by $Tf(x) = \int_{\Omega_x} K(x, y)f(y)dy$, where K is a certain type of kernel defined by the authors. The authors prove that T is locally bounded on $L^p(\Omega)$ for $1 < p < \infty$. They also obtain L^p estimates for an associated maximal function. In Phong and Stein [489] the authors consider integrals of the form $Sf(x, r) = \int_0^\infty \int_\Omega K(x, y; r, s)f(y, s)dyds$, which are called Hilbert integrals. They study Hilbert integrals arising from parametrices for the $\bar{\partial}$ -Neumann problem on a strongly pseudoconvex domain. The definition of Hilbert integrals shows that they arise from superpositions of a family of singular Radon transforms. By using the results on singular Radon transforms, they obtain estimates for solutions of the $\bar{\partial}$ -Neumann problem. In [490] Phong and Stein give a new proof of the boundedness of the operator T . This proof extends to the case where $n = 1$.

Related papers of interest are Phong and Stein [491], Phong [486], and Seeger [548].

Christ et al. [93] study a "singular" generalized k -plane transform. It is defined by $Tf(x) = \int_{M_x} f(y)K_x(y)d\sigma_x(y)$, where M_x is a smooth family of smooth k -dimensional manifolds in $\mathbb{R}^n$ with $x \in M_x$, where K_x is a k -dimensional Calderón–Zygmund kernel with singularity at x and $d\sigma_x$ is an integration measure on M_x with smooth density. They also define an associated maximal operator. Their main result is that under an appropriate curvature condition on the manifolds M_x , the singular generalized k -plane transform T is a bounded mapping from L^p to L^p . See also Greenblatt [231, 232].

Greenblatt [233] proves that translation invariant Radon transforms along certain curves are bounded operators from L^p to L^q for certain weighted L^p and L^q spaces with $p < q$.

Grinberg [236] analyzes the Radon transform on complex projective n space. He shows that the Radon transform is a multiple of the identity on irreducible representations of the space of continuous functions on $\mathbb{CP}^n$ and he evaluates these scalars. This gives an effective way of inverting the Radon transform on this space.

Sharafutdinov [554] is a booklength treatment of the theory of the x-ray transform on Riemannian spaces. This is the transform which integrates over the geodesics of a Riemannian manifold M . Sharafutdinov also considers the integration of symmetric tensor fields over such geodesics and he gives a variety of applications. It is not possible to determine a symmetric tensor of order $m > 0$ from its integrals over geodesics, because the kernel of the associated transform is not trivial even when restricted to spaces of well behaved functions. This is true even in the simple case when $M = \mathbb{R}^n$, where the geodesics are straight lines. Sharafutdinov shows that it is possible to decompose a tensor field into two parts which he terms the solenoidal part and the potential part. He shows that it is possible to recover the solenoidal part of a tensor field in the case $M = \mathbb{R}^n$, via an inversion formula. On more general Riemannian manifolds it is only possible to show uniqueness of the solenoidal part. Some of the applications are to the theory of elasticity and Maxwell's equations. Sharafutdinov introduces and studies an integral transform of tensor fields, which generalizes the x-ray transform of the earlier part of the book and which is useful in studying problems in the theory of elasticity. Of related interest is Sharafutdinov [555] and Vertgeim [609].

Branson, Ólafsson, and Schlichtkrull [66] develop the theory of the Fourier and Radon transform of vector bundle sections over certain symmetric spaces. They also obtain a proof of the Huygens principle and equipartition of energy for the Dirac and Maxwell equations.

Let H be a half-disk. Palamodov [465] shows how to reconstruct compactly supported functions on H from their integrals over half-circles that are contained in H and have their centers on the diameter of H . He does this by reducing the problem to that of inverting the ordinary Radon transform over a limited range of angles.

5.10.5 The Finite Radon Transform

Marr [425] develops an orthonormal basis of $L^2(B(0, 1))$ which is closely related to the Chebyshev polynomials and whose Radon transforms form an orthonormal basis of $L^2(Z, w)$ where w is a certain weight function. Using these bases, he gives a method of finding the best solution f in the least-squares sense to the equation $g = Rf$, where $g(\theta, \cdot)$ agrees with input data obtained from x-ray projections from finitely many equally spaced directions on the unit circle.

Logan and Shepp [394] consider the following problem: given the finite Radon transform $(R_{\theta_1} f, \dots, R_{\theta_m} f)$ of a function in $L^2(B)$, where B is the unit ball of $\mathbb{R}^2$, find a function g with "least energy" and the same finite Radon transform. Least energy means that of all the functions sharing the same finite Radon transform with f , g satisfies: $\int_B |g(x)|^2 dx$ is a minimum. They show that such a solution g exists and is a ridge function, that is, a function which is constant on lines perpendicular to some fixed direction θ . They are able to get a closed form solution in the case that the directions are equally spaced.

Let R_θ be the set of L^2 functions on the unit disk of $\mathbb{R}^2$ which are constant on each line making the angle θ with the x axis. Shapiro [553] proves that $R_{\theta_1} \oplus \cdots \oplus R_{\theta_k}$ is closed in L^2 on the unit disk of $\mathbb{R}^2$.

Fosset [177] shows that the Radon projection R_θ continuously maps the Hardy space $H^1(\mathbb{R}^n)$ onto $H^1(\mathbb{R})$. Carton-Lebrun [78] gives another derivation of this result. We note that Strichartz [584] proved that the Radon transform is a bounded operator on these Hardy spaces.

Davison [120] shows that limited-angle reconstruction using the Radon transform on $\mathbb{R}^2$ is extremely ill conditioned as the range of angles decreases from the full range (compare, Defrise and De Mol [125]).

Petersen, Smith, and Solmon [479] have some very interesting results on the closure of finite sums of subspaces of Lebesgue spaces $L^p(D)$ and Sobolev spaces $L_s^p(D)$. These results lead directly to a result on the closure and structure of the range of the finite Radon transform. The interest in the closure of the range is that it leads to continuity of the generalized inverse of the finite Radon transform.

For simplicity we state the results assuming that D has a smooth boundary. Later in this discussion we state the general conditions on D for which their results are valid. We now describe their result for the finite Radon transform, which requires some definitions. Let $Rf = (R_{\theta_1}f, \dots, R_{\theta_m}f)$ be the finite Radon transform defined on the subset $\{\theta_1, \dots, \theta_m\} \subset S^{n-1}$. For each $\theta = \theta_j$ we let $\mu_\theta(t)$ be the $(n-1)$ -dimensional Lebesgue measure of the section of D by the hyperplane $\langle x, \theta \rangle = t$, we let $L^p(\mu_\theta)$ be the set of functions defined on D such that $|g|^p \mu_\theta^{1-p}$ is integrable, and we define $L^p(\mu) = L^p(\mu_1) \oplus \cdots \oplus L^p(\mu_m)$. The authors show that $R : L^p(D) \rightarrow L^p(\mu)$ is a well-defined operator such that the following result is valid:

Theorem 5.11 ([479]). *If D is a bounded open set with smooth boundary in $\mathbb{R}^n$, then the range of $R : L^p(D) \rightarrow L^p(\mu)$ is closed and has the following structure: the range consists of all $g \in L^p(\mu)$ such that*

$$\sum_{j=1}^m \int g_j(t) h_j(t) dt = 0 \text{ whenever } \sum_{j=1}^m h_j(\langle x, \theta_j \rangle) = 0$$

where each h_j is a polynomial of degree $\leq m-2$.

This theorem depends on the following result on the closure of sums of subspaces. For this result, let $\eta_1, \dots, \eta_m$ be given linear subspaces of $\mathbb{R}^n$ and let N_j be the subspace of $L^p(D)$ consisting of functions constant on almost all translates of η_j . Then we have:

Theorem 1.4 ([479]). *If D is a bounded open set with smooth boundary in $\mathbb{R}^n$, if $1 < p < \infty$, and $\eta_j + \eta_k = \mathbb{R}^n$ if $j \neq k$, then $N_1 + \cdots + N_m$ is a closed subspace of $L^p(D)$. The same result is true if we replace $L^p(D)$ by the Sobolev space $L_s^p(D)$ where s is an integer ≥ 0 .*

The technique of proof depends on results on systems of very strongly elliptic partial differential equations which are developed in this paper by the authors. However, describing these results takes us too far afield and we refer the interested reader to [479] for the details.

These results extend to more general domains. The condition that D has a smooth boundary can be replaced by the condition that D is a Lipschitz graph domain (see the paper for the definition). The authors also give a very simple example of a domain which is not a Lipschitz graph domain and for which a finite sum of the form $N_1 + N_2$ is not closed.

Finally, we remark that there are several other interesting aspects to [479] that we choose not to present here because they do not relate directly to the Radon transform.

Boman [58] has results for the finite x-ray transform of functions in $L^p(D)$ analogous to those of the Petersen, Smith, and Solmon paper [479] for the finite Radon transform. If $n \geq 3$, then the condition $\eta_j + \eta_k = \mathbb{R}^n$ if $j \neq k$ fails if each η_j is a line. However, Boman is able to prove in this case that $N_1 + \cdots + N_m$ is closed in $L^q(D)$, where q is the conjugate index to p . He requires a condition on the nonvanishing of the principal curvature of the boundary of D . Svensson [590], using the results of Svensson [589], gives a more functional analytic proof of Boman's result.

Finch and Solmon [173] have several results on the closure of the range of the finite divergent beam transform. Let Ω be an open convex subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ with a C^2 boundary of strictly positive Gaussian curvature. Assume that the set of sources A is contained in the complement of the closure of Ω and that no line containing a pair of the sources is tangent to $\partial\Omega$. Then a special case of their main theorem shows that the image of the finite divergent beam transform acting on $L_0^p(\Omega)$ is closed in a weighted L^p space. The general case of this theorem applies to a sort of weighted divergent beam transform, see [173] for details.

Several authors had results for the range closure problem for the finite Radon transform before Petersen, Smith, and Solmon [479]. Logan and Shepp [394] proved the closure of $N_1 + \cdots + N_m$ in the case $n = 2$, the directions are equally spaced and D is a disk. Hamaker and Solmon [255] generalized this to arbitrary directions. Falconer [155] proved this result for arbitrary n , $2 \leq p \leq \infty$, convex D and where the spaces η_j have the property: For each point x in the closure of D , there is at most one η_j such that $(x + \eta_j) \cap \overline{D}$ has dimension $< n - 1$.

Bolker's Finite Radon Transform

Bolker has defined a Radon transform which is truly finite. In our definition of a finite Radon transform it is only the number of directions that is finite. Bolker's finite Radon transform is defined as follows: let $C(X)$ denote the vector space of complex valued functions defined on the set X . Let X be a finite set and let S be a subset of the power set of X . Define $R : C(X) \rightarrow C(S)$ by $Rf(y) = \sum_{x \in y} f(x)$. In analogy to the Radon transform, the points of X play the role of the points in $\mathbb{R}^n$, the subsets of S play the role of hyperplanes in $\mathbb{R}^n$, and summation plays the role of integration. Define $G_x = \{S : x \in S\}$ and let $|A|$ denote the cardinality of the set A .

In [57] Bolker finds an inversion formula for any Bolker finite Radon transform satisfying the following conditions called the Bolker assumptions.

1. $|G_x|$ is constant for $x \in X$, and
2. $|G_x \cap G_y|$ is a constant different from $|G_x|$ for all $x \neq y$ in X .

There is a microlocal analogue of the Bolker assumptions which applies to generalized Radon transforms defined by double fibrations. In this case one obtains local invertibility of the generalized Radon transform. See section 5.5 for more details.

Frankl and Graham [178] define the following special case of the Bolker Radon transform when X is a finite group. The set S consists of translates of a fixed subset $B \subset X$ under the group operation. More precisely, $S = \{aB : a \in X\}$. In the special case that $X = \mathbb{Z}_p^N$ where p is prime, Frankl and Graham are able to prove the invertibility of R_S provided that there is a maximal subgroup X_0 of X such that $|B \cap aX_0|$ is constant, independent of $a \in X$.

5.10.6 Generalizations to Finite Fields and Commutative Rings

There is some work on generalizing the Radon transform to Grassmannians over finite fields and commutative rings: Cernov [82]–[84], Graev and Pavlenko [228], Petrov [480, 481], Zelevinskii [627], Pavlenko [470], Petrov and Cernov [483, 484], Petrov [482], and Grinberg [239].

5.10.7 Other Radon Transforms

The exterior Radon transform integrates a function over all hyperplanes missing a certain ball in $\mathbb{R}^n$. This transform arises in CT when it is desired to image organs surrounding the heart. The exterior Radon transform essentially ignores any data generated by the beating heart. If it can be inverted, then the surrounding tissue can be imaged. Quinto [502] develops an inversion method for the exterior Radon transform via a singular value decomposition. Perry [472] reconstructs functions in $\mathbb{R}^2$ from the exterior Radon transform. Lissiano [391] shows that the inverse exterior Radon transform on $\mathbb{R}^2$ is not continuous into any Sobolev space, and in fact is unstable in the sense that small changes in the exterior Radon transform can correspond to large changes in the original function. This is in contrast to the situation for the ordinary Radon transform restricted to Sobolev spaces of compactly supported functions. Natterer [436] showed that in this case the inverse Radon transform is continuous.

Alliney and Sgallari [16] study the conditioning and stability of the inversion formula for the exterior Radon transform developed by Cormack [100]. Cormack's algorithm utilizes solutions to Abel type integral equations. Alliney and Sgallari show that this algorithm is ill conditioned in the sup-norm topology. They show that without regularization noise is blown up by a huge factor, but with Tikhonov regularization, the noise is reduced to a suitable level.

The spherical Radon transform integrates functions over spheres intersecting the origin in $\mathbb{R}^n$. Quinto [500] proves the injectivity of the spherical Radon transform and characterizes its range by certain moment conditions. Solmon [571] constructs an inversion formula for the spherical Radon transform on $\mathbb{R}^n$ (compare, Rhee [516, 517] for inversion results for a related transform). Hlawka [310]–[312] develops a variation of the spherical Radon transform. See also Kölzow and Singer [356]. Also of interest

are Cormack and Quinto [107], Zwaan [629], Groemer [241], Yagle [624], Rubin [527, 529], and Rubin and Ryabogin [539].

Pesenson and Grinberg [473] give an approximate inversion of the spherical Radon transform on the n -dimensional sphere together with Sobolev norm error estimates.

Carswell and Moon [77] use a generalized Radon transform to remove noise from seismic data. Seismologists are able to observe a function $V(t, x)$ of wave travel time t and position of observation x . The function V is generated by observing reflections of seismic waves. However, these observations are corrupted with noise generated by multiple reflections. It is known how to define a generalized Radon transform $\mathcal{R}$ such that $\mathcal{R}V$ is a function of intercept time τ and wave slowness p (Chapman [86]; Carswell and Moon [77]). Although it is not possible to detect the noise generated by multiple reflections by observing the function V in the (t, x) plane, this Radon transform has the property that desirable single reflections transform to ellipses in the (τ, p) plane, whereas undesirable multiple reflections transform to nonclosed curves. Therefore, the multiple reflections are easily eliminated in the Radon domain. Taking the inverse Radon transform of the resulting data then yields the desired function V free of multiple reflections (compare, Schuster [545]).

Denisjuk [127] considers generalized k -plane transforms which integrate functions over families of k -dimensional submanifolds of $\mathbb{R}^n$. These families must satisfy a completeness condition which is explicitly described in [127]. With this much generality, one cannot expect explicit inversion formulas. However, Denisjuk is able to give an inversion formula whose main term is similar to the classical inversion formula for the k -plane transform on $\mathbb{R}^n$ together with an error term involving a pseudodifferential operator of order -1 . In the case $k = 1$, Denisjuk gives conditions which generalize the Kirillov and Tuy conditions (see definition 5.7).

Fridman [179] investigates uniqueness for a certain family of generalized Radon transforms on the unit square of $\mathbb{R}^2$.

Bailey et al. [30] use complex analytic methods to invert the Funk transform; they do so by solving a $\bar{\partial}$ problem on $\mathbb{CP}^2$.

5.11 Appendix

In this appendix we provide the proof of the lemma 5.6.

Lemma 5.6. *Let $\mathfrak{S}$ be an $n - 1$ sphere of radius r , let x be a point in $\mathbb{R}^n$ which is inside $\mathfrak{S}$ and let f be an integrable function on S^{n-1} . Then*

$$\int_{S^{n-1}} f(\theta) d\theta = \frac{1}{r} \int_{\mathfrak{S}} f\left(\frac{a-x}{|a-x|}\right) |a-x|^{-n} |\langle a, a-x \rangle| da$$

where $d\theta$ is Lebesgue measure on the unit sphere S^{n-1} and da is Lebesgue measure on the sphere $\mathfrak{S}$.

Proof. We can assume without loss of generality that the center of $\mathfrak{S}$ is the origin. Define $\gamma : \mathfrak{S} \rightarrow S^{n-1}$ by

$$\gamma(a) = \frac{a - x}{|a - x|}.$$

Also define maps $\phi : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$ by

$$\phi(y) = \frac{y}{|y|}$$

and $\eta : \mathfrak{S} \rightarrow \mathbb{R}^n \setminus \{0\}$ by

$$\eta(a) = a - x.$$

Then γ factors into $\gamma = \phi \circ \eta$ and the pullback γ^* of γ to differential forms has the factorization

$$\gamma^* = \eta^* \circ \phi^*. \quad (5.60)$$

We now calculate $\gamma^*(d\theta)$, where $d\theta$ is the $n - 1$ form which is the volume element of S^{n-1} and which has the formula

$$d\theta = \sum_{j=1}^n (-1)^{j+1} y_j dy(\widehat{j})$$

(Flanders [176]). The symbol $dy(\widehat{j})$ means the exterior product $dy_1 \wedge \cdots \wedge dy_n$ with dy_j missing. Related to this form is the $n - 1$ form

$$\Theta = \sum_{j=1}^n (-1)^{j+1} y_j dy(\widehat{j})$$

defined on $\mathbb{R}^n$. Flanders [176], page 76, has an elegant proof that $\phi^*(d\theta) = \frac{1}{|y|^n} \Theta$. From this and equation (5.60) we get

$$\gamma^*(d\theta) = \eta^* \left(\frac{1}{|y|^n} \Theta \right) = \sum_{j=1}^n (-1)^{j+1} \eta^* \left(\frac{y_j}{|y|^n} \right) \eta^*(dy(\widehat{j})) . \quad (5.61)$$

But the pullback of the coefficient $\eta^*\left(\frac{y_j}{|y|^n}\right)$ is computed by substituting $\eta(a) = a - x$ for the variable y . Also $\eta^*(dy(\widehat{j}))$ is the $n - 1$ form on the sphere $\mathfrak{S}$ which corresponds to $dy(\widehat{j})$ restricted to the tangent space of $\mathfrak{S}$. It is known (Spivak [580] in dimension $n = 3$, but easily generalized to n dimensions) that on a hypersurface M in $\mathbb{R}^n$, with volume element $d\mu$ we have $dy(\widehat{j}) = (-1)^{j+1} \nu_j d\mu$, where ν is the unit outward normal. In the case that $M = \mathfrak{S}$, $d\mu = da$ this gives

$$\eta^*(dy(\widehat{j})) = (-1)^{j+1} \frac{a_j}{r} da.$$

Substituting this into equation (5.61) gives

$$\begin{aligned}\gamma^*(d\theta) &= \sum_{j=1}^n (-1)^{j+1} \frac{a_j - x_j}{|a - x|^n} \left((-1)^{j+1} \frac{a_j}{r} da \right) \\ &= \frac{1}{r |a - x|^n} \sum_{j=1}^n (a_j - x_j) a_j da = \frac{\langle a - x, a \rangle}{r |a - x|^n} da\end{aligned}$$

Recalling that a is a variable point on a sphere of radius r and that $|x| < r$, it follows from the Cauchy–Schwarz inequality that $\langle a - x, a \rangle \geq 0$ and therefore we may insert absolute values to get

$$\gamma^*(d\theta) = \frac{|\langle a - x, a \rangle|}{r |a - x|^n} da.$$

More generally, if f is an integrable function on the sphere, then

$$\gamma^*(f(\theta) d\theta) = f\left(\frac{a - x}{|a - x|}\right) \frac{|\langle a - x, a \rangle|}{r |a - x|^n} da.$$

Integrating these forms yields the proof. ■

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Index of Notations and Symbols

■	denotes the end of a proof, page 63	$\mathcal{D}$	divergent beam x-ray transform, page 279
Δf	Laplacian, $\Delta f = \sum \frac{\partial^2 f}{\partial x_j^2}$, page 168	$\mathbb{D}^\alpha f$	Riesz fractional derivative, page 208
δ	Dirac δ distribution, page 133	$\mathcal{D}'_{L^1}$	space of integrable distributions, page 118
$\delta(K)$	diameter of the set K , page 129	$\mathcal{D}_s(\mathcal{G}_{k,n})$	page 188
Λ^α	lambda operator, page 164	$\mathcal{D}_s(\mathbb{R}^n)$	page 188
∂^α	partial derivative operator in multi-index notation, page 130	$dx(\hat{j})$	denotes the $n - 1$ form $dx_1 \wedge \dots \wedge dx_n$ with dx_j missing, page 134
Φ	Lizorkin class of functions in S with vanishing moments, page 207	dx'	Lebesgue measure on a k space, e.g., $dx_1 \dots dx_{n-1}$, page 69
Θ	operator which maps the function $g(\theta, p)$ on $S^1 \times \mathbb{R}$ to $\Theta g(\theta, p) = (\theta_1 + i\theta_2)g(\theta, p)$, page 315	$\mathcal{F}(f)$	Fourier transform of the function f . Is equal to $\hat{f}$, page 85, 131
$\nu_{k,n}$	Haar measure on the Grassmannian $G_{k,n}$, page 142	$\mathcal{F}^{-1}(f)$	inverse Fourier transform of the function f . Is equal to $\tilde{f}$, page 131
$\theta^\perp$	denotes the linear space orthogonal to θ , page 63	$f * g$	convolution, page 83
$\rho_{k,n}$	Coefficient of the Riesz kernel, page 172	$f _S$	restriction of a function f to a set S , page 129
$ A $	Lebesgue measure of the set A , page 65	$\langle f, g \rangle_F$	inner product on the function space F , page 130
$B^n(c, r)$	ball of center c , radius r in $\mathbb{R}^n$, page 129	f_a	translation of the function f by the vector a , $f_a(x) = f(x - a)$, page 70
$B(c, r)$	page 129	$f_j \xrightarrow{L^p} f$	convergence in L^p - norm, page 129
$B^n(r)$	ball of center 0, radius r in $\mathbb{R}^n$, page 129	$\ f\ _{L^p}$	L^p norm of the function f , page 129
$B(r)$	page 129	$f(x, y)$	value of the function f at x, y , page 16
$C^0(X)$	space of continuous functions on X , page 129	$\hat{f}(x, y)$	Fourier transform of the function f . Is equal to $\mathcal{F}(f)$, page 85, 131
$C^m(X)$	m times continuously differentiable functions on the set X , page 129	$\tilde{f}$	inverse Fourier transform of the function f . Is Equal to $\mathcal{F}^{-1}(f)$, page 131
C_0^∞, L_0^p	notation for compactly supported functions, page 131	$\mathcal{G}_{k,n}$	affine Grassmannian, page 150
C_v^∞	space of differentiable functions vanishing at infinity, page 118	$G_{k,n}$	Grassmann manifold, page 136
$\mathbb{C}$	complex number system, page 59	$GL(n)$	general linear group, page 138
$\mathbb{C}^n$	complex Euclidean n space, page 129	$\mathbb{H}^2$	hyperbolic disk, page 336
$\mathbb{CP}^n$	complex projective space, page 339	$H\ddot{o}l_\delta$	class of Hölder continuous functions of order δ , page 313

I_α	Riesz potential, page 172	R_α	Riesz kernel, page 172
$Jg(x)$	Jacobian of a differentiable function = $ \det(g'(x)) $, page 69	$R_\theta^\# g$	backprojection in one direction, page 28
L^∞	Lebesgue space of essentially bounded measurable functions, page 129	$R_\theta f(s)$	value of the Radon transform of f at θ, s , page 19
L^p	Lebesgue space of L^p functions, page 129	$Rf(\theta, s)$	value of the Radon transform of f at θ, s , same as $R_\theta f(s)$, page 19
L_{loc}^p	space of locally L^p functions, page 131	$\mathbb{R}^n$	Euclidean n space, page 59
$O(n)$	orthogonal group, page 139	$\mathcal{S}$	Schwartz class of rapidly decreasing functions, page 130
$\overline{P}$	extended k -plane transform, page 190, 191	$ S^{n-1} $	volume of the unit sphere in $\mathbb{R}^n$, page 129
$P^\#$	formal adjoint of the k -plane transform = backprojection operator, page 152	S^{n-1}	unit sphere, page 61
$P_\mu f$	attenuated x-ray transform with weight μ , page 291	$\text{supp}(f)$	support of the function f , page 129
$P_\eta f$	k -plane projection in direction η , page 151	$T_\mu f$	exponential Radon transform with weight μ , page 294
Pf	k -plane transform or k -dimensional Radon transform of f , page 151	$ x $	norm in $\mathbb{R}^n$, page 60
$Pf(\eta, x'')$	k -plane transform, value of, page 151	$x \cdot y$	synonym for inner product on $\mathbb{R}^n$, equal to $\langle x, y \rangle$, page 60
proj_η	orthogonal projection onto the subspace η of $\mathbb{R}^n$, page 152	$\langle x, y \rangle$	inner product on $\mathbb{R}^n$, equal to $x \cdot y$, page 60
$\mathbb{R}$	real number system, page 59	x'	$x' = (x_1, \dots, x_{n-1})$, page 69
$\mathcal{R}_\mu$	generalized Radon transform with weight μ , page 300	x', x''	if x is a variable in $\mathbb{R}^n$ and $\eta \in G_{k,n}$, then $x = x' + x''$, where $x' = \text{proj}_\eta x$ and $x'' = \text{proj}_{\eta^\perp} x$, page 137
R	Radon transform, page 73	Xf	x-ray transform of f , page 151
$R^\#$	backprojection operator, page 99	$X_\mu f$	exponential x-ray transform with weight μ , page 295
$R_\mu f$	attenuated Radon transform with weight μ , page 292	$X_\theta f$	x-ray projection in direction θ , page 152
$R_\theta f$	Radon projection, page 73	Z^n	cylinder of dimension n , $Z^n = S^{n-1} \times \mathbb{R}$, page 73

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